RESTRICTION OF ISOTopies

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Communicated by Deane Montgomery, August 23, 1962

Let $M$ be a connected and simply connected topological manifold (with or without boundary) and $m$ a fixed point in Int $M$, the interior of $M$. Let $h_0$ and $h_1$ be two isotopic homeomorphisms of $M$, each of which leaves $m$ fixed.

It is the object of this note to show that, under these conditions, $h_0/M - m$ and $h_1/M - m$ are isotopic homeomorphisms of $M - m$.

With the standard definition of isotopy, the result follows immediately from the covering homotopy theorem, but with a somewhat more liberal (and frequently more natural) definition of isotopy, the argument is less direct. In fact in this case I can obtain the result only with the aid of the apparently irrelevant assumption that $M$ can support a piecewise linear structure.

The converse question of extending isotopies on a space to isotopies on its one-point compactification has already been answered affirmatively by R. H. Crowell [1] in the much more general setting of locally compact Hausdorff spaces.

1. Definitions. If $h_0$ and $h_1$ are homeomorphisms of $X$ onto $Y$, an isotopy between $h_0$ and $h_1$ is a continuous map

$$H: X \times [0, 1] \to Y \times [0, 1]$$

such that

(i) $H(x, 0) = (h_0(x), 0)$ for all $x \in X$,
(ii) $H(x, 1) = (h_1(x), 1)$ for all $x \in X$,
(iii) $H/XXt$ is a homeomorphism of $X \times t$ onto $Y \times t$ for all $t \in [0, 1]$.

It is shown in [1] that if $X$ is a locally compact Hausdorff space, then condition (iii) above implies

(iii') $H$ is a homeomorphism.

$H$ is called a weak isotopy between $h_0$ and $h_1$ if $H$ satisfies conditions (i), (ii) and (iii'). Thus if $X$ is locally compact and Hausdorff (in particular, if $X$ is a manifold), an isotopy is also a weak isotopy, so that isotopic homeomorphisms will also be weakly isotopic.

Weak isotopy is an important notion in the study of topological manifolds. For example, the extendability of a homeomorphism de-

1 The author holds a National Academy of Sciences Postdoctoral Research Fellowship.
fined on the boundary of a manifold to a homeomorphism of the whole manifold depends only on the weak isotopy class of the homeomorphism.

If $M$ is a connected topological manifold and $m$ a fixed point in $\text{Int } M$, then $H(M)$ will denote the topological group of homeomorphisms of $M$ under the compact-open topology, and $H(M, m)$ the closed subgroup of homeomorphisms which leave $m$ fixed.

The projection of $M \times [0, 1]$ onto $M$ will be denoted by $pr_M$. If $H$ is a weak isotopy between two homeomorphisms of $(M, m)$ then the curve

$$\gamma: [0, 1] \to M,$$

defined by $\gamma(t) = pr_M(H(m, t))$, is a closed curve in $M$ based at $m$, which we call the trace of $H$.

2. Restriction of isotopies.

**Theorem 2.1.** Let $M$ be a connected and simply connected topological manifold and $m$ a fixed point in $\text{Int } M$. If $h_0$ and $h_1$ are two isotopic homeomorphisms of $M$, each of which leaves $m$ fixed, then $h_0/M - m$ and $h_1/M - m$ are isotopic homeomorphisms of $M - m$.

Let

$$H: M \times [0, 1] \to M \times [0, 1]$$

be an isotopy between $h_0$ and $h_1$, and let

$$h_t: M \to M$$

be the homeomorphism of $M$ defined by

$$H(x, t) = (h_t(x), t).$$

The following facts are well known.

(i) The map $\Gamma: [0, 1] \to H(M)$, defined by $\Gamma(t) = h_t$, is continuous.

(ii) $H(M)$ is a principal bundle over $\text{Int } M$ with fibre and group $H(M, m)$ and projection $p: H(M) \to \text{Int } M$ defined by $p(h) = h(m)$.

Then $\gamma = p\Gamma$, the trace of the isotopy $H$, is contractible because $M$ is simply connected. Hence by the covering homotopy theorem, $\Gamma'$ can be deformed into a path $\Gamma'$ which connects $h_0$ with $h_1$ and lies entirely in the fibre $p^{-1}(m) = H(M, m)$. Then

$$H': M \times [0, 1] \to M \times [0, 1],$$

defined by

$$H'(x, t) = (\Gamma'(t)(x), t),$$

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is an isotopy between \( h_0 \) and \( h_1 \) such that, for all \( t \in [0, 1] \),

\[ H'(m, t) = (m, t). \]

Hence \( H'/((M-m) \times [0, 1]) \) is an isotopy between \( h_0/M-m \) and \( h_1/M-m \).

3. **Homma's theorem.** Homma [2] has recently proved the following theorem, in the statement of which, \( U_\varepsilon(\bar{P}^k) \) denotes the set of points whose distance from \( \bar{P}^k \) is less than \( \varepsilon \).

**HOMMA’S THEOREM.** Let \( M^n, \bar{M}^n \) and \( \bar{P}^k \) be two finite combinatorial \( n \)-manifolds and a finite polyhedron such that \( \bar{M}^n \) is topologically embedded in \( M^n \), \( \bar{P}^k \) is piecewise linearly embedded in \( \text{Int} \bar{M}^n \) and \( 2k+2 \leq n \). Then for any \( \varepsilon > 0 \), there is an \( \varepsilon \)-homeomorphism \( F \) of \( M^n \) onto \( M^n \) such that

\[ F/M^n - U_\varepsilon(\bar{P}^k) = 1, \]

\[ F/\bar{P}^k \text{ is piecewise linear}. \]

Combining the reciprocal approximation technique employed by Homma to prove the above theorem with Lemma 2 of [2], one easily obtains the following result, which may be regarded as an indirect corollary to Homma’s theorem.

**THEOREM 3.1.** Let \( M^n \) be a topological \( n \)-manifold with boundary \( B^{n-1} \). Let \( M_1^n \) and \( M_2^n \) be two combinatorial \( n \)-manifolds, each of which has \( M^n \) for underlying space. Let \( P_1 \) be a polygonal arc in \( M_1^n \) which meets \( B_1^{n-1} \) only at its endpoints. If \( n \geq 4 \), then for any \( \varepsilon > 0 \) there is an \( \varepsilon \)-homeomorphism \( F: M_1^n \to M_2^n \) such that

\[ F/M_1^n - U_\varepsilon(P_1) = 1, \]

\[ F/B_1^{n-1} = 1, \]

\( F/P_1 \) is piecewise linear.

4. **Restriction of weak isotopies.**

**THEOREM 4.1.** Let \( M \) be a connected and simply connected topological manifold which can support a piecewise linear structure, and \( m \) a fixed point in \( \text{Int} M \). If \( h_0 \) and \( h_1 \) are two weakly isotopic homeomorphisms of \( M \), each of which leaves \( m \) fixed, then \( h_0/M-m \) and \( h_1/M-m \) are weakly isotopic homeomorphisms of \( M-m \).

Since \( M \) can support a piecewise linear structure, triangulate \( M \times [0, 1] \) as a combinatorial manifold in which \( m \times [0, 1] \) appears as a subcomplex. Let
be a weak isotopy between \( h_0 \) and \( h_1 \). The plan is to first find a homeomorphism \( F \) of \( M \times [0,1] \) onto itself such that
\[
F/(M \times 0) \cup (M \times 1) = 1,
\]
\[
FH(m \times [0,1]) \text{ is polygonal},
\]
and then a homeomorphism \( F' \) of \( M \times [0,1] \) onto itself such that
\[
F'/(M \times 0) \cup (M \times 1) = 1,
\]
\[
F'FH(m \times [0,1]) = m \times [0,1].
\]
Then \( F'FH \) will be a weak isotopy of \( h_0 \) with \( h_1 \) which takes \( m \times [0,1] \) onto itself, and hence \( F'FH/(M-m) \times [0,1] \) will be a weak isotopy of \( h_0/M-m \) with \( h_1/M-m \).

If \( \dim M = 1 \), \( M \) is homeomorphic to an open, half-closed or closed arc, and the theorem is trivially true.

If \( \dim M = 2 \), suppose first that \( M \) is homeomorphic to \( S^2 \). The existence of both \( F \) and \( F' \) is demonstrated in §9 of [3]. If \( M \) is not homeomorphic to \( S^2 \), then \( \text{Int} M \) is homeomorphic to Euclidean 2-space, \( \mathbb{R}^2 \). The existence of \( F \) is shown in §9 of [3], while the existence of \( F' \) follows from a standard argument involving Dehn’s lemma [4] and the fact that an orientation preserving homeomorphism of a 2-sphere is isotopic to the identity.

If \( \dim M \geq 3 \), let \( M^a \) be \( M \times [0,1] \) triangulated as above, and let \( M^a_1 \) be \( M \times [0,1] \) with the triangulation induced from \( M^a_2 \) by the homeomorphism \( H \). Since \( m \times [0,1] \) appears as a subcomplex of \( M^a_2 \), \( H(m \times [0,1]) \) appears as a subcomplex of \( M^a_1 \). Letting \( P_1 = H(m \times [0,1]) \), the existence of \( F \) is assured by Theorem 3.1.

Since \( M \times [0,1] \) is simply connected, the polygonal arc \( FH(m \times [0,1]) \) is homotopic to the polygonal arc \( m \times [0,1] \) in \( M \times [0,1] \). Since \( \dim (M \times [0,1]) \geq 4 \), a general position argument will produce \( F' \).

5. An application. Think of \( S^n \) as the one-point compactification of \( \mathbb{R}^n \) by the point \( \infty \). Then the following may be regarded as a corollary to Theorem 4.1.

**Theorem 5.1.** If \( h \) is a homeomorphism of \( (S^n, \infty) \) which is weakly isotopic to the identity, then \( h/\mathbb{R}^n \) is weakly isotopic to the identity homeomorphism of \( \mathbb{R}^n \).

For the theorem is trivial when \( n = 1 \) and \( S^n \) is simply connected when \( n > 1 \).
Now let \( h \) be a homeomorphism of \((S^n, \infty)\), and from \( S^n \times [0, 1] \) form a space \( M \) by identifying \((x, 0)\) with \((h(x), 1)\) for each \( x \in S^n \). Let \( \phi: S^n \times [0, 1] \rightarrow M \) be the decomposition map.

**Theorem 5.2.** If \( M \) is homeomorphic to \( S^n \times S^1 \), then \( \phi(R^n \times [0, 1]) \) is homeomorphic to \( R^n \times S^1 \).

If \( M \) is homeomorphic to \( S^n \times S^1 \), then it follows from [5] that \( h \) must be weakly isotopic to the identity. By the preceding theorem, \( h/R^n \) must also be weakly isotopic to the identity, from which it easily follows that \( \phi(R^n \times [0, 1]) \) is homeomorphic to \( R^n \times S^1 \).

6. **Further results.** Theorem 2.1 is actually a special case of a more general result, which is briefly described below.

Let \( M \) be a connected manifold and \( m \in \text{Int } M \). Let \( h \) be a homeomorphism of \( M \) leaving \( m \) fixed, which is isotopic to the identity homeomorphism, \( 1_M \). Define the **trace class**, \( \tau(h) \), to be the set of all elements of \( \pi_1(M, m) \) which can be represented by traces of isotopies of \( 1_M \) with \( h \). Then \( \tau(1_M) \) is a central (and hence normal) subgroup of \( \pi_1(M, m) \), and \( \tau(h) \) is a coset of \( \tau(1_M) \). Thus \( \tau(h) \) may also be regarded as an element of the **trace group**

\[
T(M, m) = \pi_1(M, m)/\tau(1_M).
\]

Now, if \( h_0 \) and \( h_1 \) are isotopic homeomorphisms of \( M \), each of which leaves \( m \) fixed, then \( h_0^{-1}h_1 \) is isotopic to \( 1_M \), hence \( \tau(h_0^{-1}h_1) \) is defined. It then follows easily from the covering homotopy theorem applied to the bundle \( H(M) \) over \( \text{Int } M \) that \( h_0/M - m \) and \( h_1/M - m \) are isotopic homeomorphisms of \( M - m \) if and only if \( \tau(h_0^{-1}h_1) = \tau(1_M) \).

This condition is automatically satisfied when \( M \) is simply connected, hence Theorem 2.1.

Theorem 4.1 follows from a similar result about weak isotopy.

**References**


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