STABLE HOMEOMORPHISMS CAN BE APPROXIMATED
BY PIECEWISE LINEAR ONES

BY E. H. CONNELL

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A homeomorphism $h$ of $E^n$ or $S^n$ onto itself is stable if $\exists$ homeomorphisms $h_1, h_2, \ldots, h_m$ and nonvoid open sets $U_1, U_2, \ldots, U_m$ such that $h = h_mh_{m-1}\cdots h_1$ and $h_i\big|_{U_i} = I$ for $i = 1, 2, \ldots, m$. All orientation preserving homeomorphisms of $E^n$ or $S^n$ are stable provided $n = 1, 2, \text{ or } 3$. There is no example known in any dimension of an orientation preserving homeomorphism which is not stable. In fact, the conjecture that all orientation preserving homeomorphisms of $E^n$ or $S^n$ are stable is equivalent to the annulus conjecture (see [3]).

It is known that any homeomorphism of $E^n$ onto itself can be approximated by a piecewise linear one (see [2] or [6]). The purpose of this paper is to announce that if $n \geq 7$ and $h$ is a stable homeomorphism of $E^n$ or $S^n$ onto itself, then $h$ can be approximated by a piecewise linear homeomorphism, and also, in the case of $E^n$, by a diffeomorphism.

The set of all homeomorphisms on $E_n$ or $S_n$ forms a group under composition and the subset of stable homeomorphisms forms a normal subgroup. The stable group on $S_n$ is simple while the stable group on $E_n$ is not. Due to this fact, there is a shorter proof in the case of $S_n$ than in the case of $E_n$, and it is this proof which will be outlined here. The author thanks John Stallings for his assistance.

**NOTATION.** $E_n$ is Euclidean $n$-space, $S_{n-1}$ is the unit sphere in $E_n$, and $O_n$ is the open unit ball in $E_n$. Thus $O_n \cup S_{n-1} = \overline{O_n}$. For a given integer $n$, $O_n$ will usually be denoted by $O$. If $U \subseteq E_n$ and $a > 0$, $aU = \{ x \in E_n : \exists y \in U \text{ such that } x = ay \}$. $C(aU)$, the compliment of $aU$, will be denoted by $Ua$. Thus, for a given $n$, $aO$ will be the canonical open ball in $E_n$ of radius $a$. If $x, y \in E_n$, $|x - y|$ will be the usual distance from $x$ to $y$. If $O$ is the origin and $x \neq O \neq y$, then $\theta \{ x, y \}$ will represent the angle in radians between the two line intervals, one joining $O$ to $x$ and the other joining $O$ to $y$. Thus $0 \leq \theta \{ x, y \} \leq \pi$. A piecewise linear structure (p.w.l. structure) or combinatorial structure on an open subset of $E_n$ or $S_n$ is a triangulation such that the star of each vertex is a combinatorial cell (see §3 of [10]). The identity function will be denoted by $I$.

The results of this paper are based primarily on Lemma 1 below, a modification of the Engulfing Lemma (see §3.4 of [10]). The proof is omitted.
Lemma 1. Suppose \( E_n (n \geq 4) \) has an arbitrary p.w.l. structure \( T \), \( K \) is a finite subcomplex of \( T \), \( \dim K \leq n - 4 \), \( a \), \( b \), and \( \epsilon \) are nos. with \( 0 < a < b \), \( \epsilon > 0 \) and \( K \subset bO = bO_n \). Then \( \exists \) a homeomorphism \( h: E_n \rightarrow E_n \) such that \( h \) is p.w.l. relative to \( T \), \( h| (a - \epsilon)O = I \), \( h| (b + \epsilon)O = I \), \( h(aO) \supset K \) and \( \theta\{h(x), x\} < \epsilon \) for \( x \in E_n \).

The proof of Lemma 2 below follows from Lemma 1 and trivial modifications of §4 of [10] and §8.1 of [11]. The proof is omitted.

Lemma 2. Suppose \( E_n (n \geq 7) \) has an arbitrary p.w.l. structure \( T \), and \( a \), \( b \), and \( \epsilon \) are nos. with \( 0 < a < b \) and \( \epsilon > 0 \). Then \( \exists \) a homeomorphism \( h: E_n \rightarrow E_n \) such that \( h \) is p.w.l. relative to \( T \), \( h| (a - \epsilon)O = I \), \( h| (b + \epsilon)O = I \), \( h(aO) \supset K \) and \( \theta\{h(x), x\} < \epsilon \) for \( x \in E_n \).

Definition. A homeomorphism \( h: S_n \rightarrow S_n \) is said to have property \( P \) if for any p.w.l. structure \( T \) on \( S_n \) and any \( \epsilon > 0 \), \( \exists \) a homeomorphism \( f: S_n \rightarrow S_n \) such that \( f \) is p.w.l. relative to \( T \) and \( |h(x) - f(x)| < \epsilon \) for \( x \in S_n \). Let \( G_n \) be the set of all homeomorphisms on \( S_n \) which possess property \( P \).

Observation A. \( G_n \) is a normal subgroup of the group of all homeomorphisms under composition.

Proof. The proof that it is a subgroup is immediate. It will be shown that \( G_n \) is normal. Suppose \( h \in G_n \) and \( g: S_n \rightarrow S_n \) is any homeomorphism. Show that \( g^{-1}hg \in G_n \). Let \( T \) and \( \epsilon \) be given.

There exists a \( \delta > 0 \) such that if \( |x - y| < \delta \), then \( |g^{-1}(x) - g^{-1}(y)| < \epsilon \). Let \( T_1 \) be the p.w.l. structure on \( S_n \) which is the \( g \) image of \( T \), \( T_1 = g(T) \). Thus if \( v \) is a simplex of \( S_n \) in the triangulation \( T \), then \( g(v) \) is a simplex of \( S_n \) in the triangulation \( T_1 \). Since \( h \in G_n \), \( \exists \) a homeomorphism \( f: S_n \rightarrow S_n \) which is p.w.l. relative to \( T_1 \) and with \( |h(x) - f(x)| < \delta \) for \( x \in S_n \). Thus \( |g^{-1}hg(x) - f(x)| < \epsilon \) for \( x \in S_n \). Note that \( g^{-1}fg \) is p.w.l. relative to \( T \) because: \( g \) is p.w.l. from \( T \) to \( T_1 \), \( f \) is p.w.l. from \( T_1 \) to \( T_1 \) and \( g^{-1} \) is p.w.l. from \( T_1 \) to \( T \). This justifies Observation A.

Theorem 1. Let \( T \) be an arbitrary p.w.l. structure on \( S_n (n \geq 7) \) and let \( h: S_n \rightarrow S_n \) be a stable homeomorphism. If \( \epsilon > 0 \), \( \exists \) a homeomorphism \( f: S_n \rightarrow S_n \) such that \( f \) is p.w.l. relative to \( T \) and \( |h(x) - f(x)| < \epsilon \) for \( x \in S_n \).

Proof. The set of all stable homeomorphisms of \( S_n \) is a simple, normal subgroup of the group of all homeomorphisms. The fact that it is a normal subgroup is trivial and the fact that it is simple follows from [1] and is even stated explicitly in Theorem 14 of [4]. Therefore, using Observation A, it will follow that \( G_n \) contains the stable group if \( G_n \) contains some stable homeomorphism distinct from the
identity. This will now be shown.

Let \( h \) be a symmetric radial expansion, i.e., let \( h: E_\mathbb{R}^n \to E_\mathbb{R}^n \) be a homeomorphism such that \( h(x) = x \) for \( ||x|| \geq 1 \), \( h(0) = 0 \), \( \theta \{ h(x), x \} = 0 \) for all \( x \), and if \( 0 < r < 1 \), \( \exists a \, no. \ u(r), r < u(r) < 1 \) such that \( h[r(0 - O)] = u(r)(0 - O) \). Let \( T \) be any p.w.l. structure on \( E_\mathbb{R}^n \) and \( \varepsilon > 0 \). It will be shown that \( \exists f: E_\mathbb{R}^n \to E_\mathbb{R}^n \) which is a p.w.l. homeomorphism relative to \( T \) and with \( \epsilon(x) = x \) for \( ||x|| \geq 1 \) and \( \| h(x) - f(x) \| < \varepsilon \) for \( x \in E_\mathbb{R}^n \). Since \( h \) determines a homeomorphism from \( S_n \) to itself by defining \( \bar{h}(\infty) = \infty \), this will show that \( G_n \) is nontrivial and will complete the proof of Theorem 1.

Let \( \epsilon > \epsilon_0 < r_1 < r_2 \cdots < r_{m+1} = 1 \) be nos. such that \( \{ u(r_{i+2}) - u(r_i) \} < \epsilon/2 \) for \( i = 0, 1, 2, \ldots, (m-1) \). By Lemma 3, \( \exists p.w.l. \) homeomorphisms \( f_1, f_2, \cdots, f_m \) such that \( f_i|_{r_{i-1}O} = I, f_i|_{Ou(r_{i+1})} = I, \theta \{ f_i(x), x \} < \epsilon/4 \) for \( x \in E_\mathbb{R}^n \), and \( f_i(r_iO) \supset u(r_i)O \) for \( i = 1, 2, \cdots, m \). Let \( f = f_1 f_2 \cdots f_m \). Now \( \bar{f} \) is a homeomorphism of \( E_\mathbb{R}^n \) onto \( E_\mathbb{R}^n \) that is p.w.l. relative to \( T \) and \( \| f(x) - h(x) \| < \epsilon \) for \( x \in E_\mathbb{R}^n \). Let \( x \in r_{k+1}O \cap O r_k = r_{k+1}O - r_kO \), \( 0 \leq k \leq m \). Then \( f(x) = f_1 f_2 \cdots f_{k+1} \) because \( f_i|_{r_{i+1}O} = I \) for \( i > k+1 \). In fact, \( f(x) = f_1 f_{k+1}(x) \) because \( f_1 f_{k+1}(x) \in Ou(r_k) \) and \( f_1|_{Ou(r_k)} = I \) for \( i < k \). (In the special case \( k = 0 \), \( f(x) = f_1(x) \).) Now since \( f(x) \) and \( h(x) \in u(r_{k+2})O \cap Ou(r_k) \), \( ||f(x)|| \) and \( ||h(x)|| \) differ by \( \epsilon/2 \). Since \( \theta \{ h(x), f(x) \} < \epsilon/2 \) is measured in radians and any radius under consideration is \( < 1 \), it follows that \( ||h(x) - f(x)|| < \epsilon \). This completes the proof.

**Theorem 2.** Let \( T \) be an arbitrary p.w.l. structure on \( E_\mathbb{R}^n (n \geq 7) \). If \( h: E_\mathbb{R}^n \to E_\mathbb{R}^n \) is a stable homeomorphism and \( \epsilon(x): E_\mathbb{R}^n \to (0, \infty) \) is a continuous function, then \( \exists a \) homeomorphism \( f: E_\mathbb{R}^n \to E_\mathbb{R}^n \) such that \( f \) is p.w.l. relative to \( T \) and \( |f(x) - h(x)| < \epsilon(x) \) for \( x \in E_\mathbb{R}^n \).

Since the stable group on \( E_\mathbb{R}^n \) is not simple, the trick used in the proof of Theorem 1 cannot be used. A direct construction of \( f \) is required. The proof is omitted.

**Theorem 3.** Suppose \( D \) is any \( C^2 \) differentiable structure on \( E_\mathbb{R}^n (n \geq 7) \). If \( h: E_\mathbb{R}^n \to E_\mathbb{R}^n \) is a stable homeomorphism and \( \epsilon(x): E_\mathbb{R}^n \to E_\mathbb{R}^n \) is a continuous function, then \( \exists a \) homeomorphism \( f: E_\mathbb{R}^n \to E_\mathbb{R}^n \) which is a \( C^2 \) diffeomorphism relative to \( D \) and such that \( |f(x) - h(x)| < \epsilon(x) \) for \( x \in E_\mathbb{R}^n \).

**Proof.** Let \( T \) be a \( C^2 \) triangulation of \( E_\mathbb{R}^n \) which is compatible with \( D \) (see [5] or [13]). By Theorem 2, \( h \) may be approximated by a homeomorphism \( f_1 \) which is p.w.l. relative to \( T \). Now by Theorems 5.7 and 6.2 of [7], \( f_1 \) may be approximated by a diffeomorphism \( f \). This completes the proof. The theorem remains true if \( C^2 \) is replaced by \( C^\infty \).
It is not clear whether or not Theorem 3 remains true when $E_n$ is replaced by $S_n$. It is known that $E_n$ and $S_n$ have to be considered as separate cases. For instance, any two differentiable structures on $E_n$ are equivalent (except possibly for $n = 4$) while this is not true on $S_n$ (see [10] and [5] respectively).

Let $D_1$ and $D_2$ be two differentiable structures on $E_n$ ($n \geq 7$). Let $h: E_n \to E_n$ be a diffeomorphism mod $D_1$. Since diffeomorphisms are always stable, according to Theorem 3, $h$ can be approximated by $f$, a diffeomorphism mod $D_2$. This type of question might be interesting on $S_n$. For instance, let $n = 7$ and $D_1$ be the ordinary differentiable structure on $S_7$ and $D_2$ be one of Milnor's bad differentiable structures. Can diffeomorphisms mod $D_1$ be approximated by diffeomorphisms mod $D_2$? They can be approximated by p.w.l. ones by Theorem 1.

If $T_1$ and $T_2$ are two p.w.l. structures on $E_n$ (or $S_n$), then any homeomorphism p.w.l. relative to $T_1$ can be approximated by one p.w.l. relative to $T_2$. This follows from Theorem 2 (resp. Theorem 1) and the fact that p.w.l. homeomorphisms are stable.

References

2. R. H. Bing, An alternate proof that 3-manifolds can be triangulated, Ann. of Math. (2) 69 (1959), 37–65.