STABLE HOMEOMORPHISMS CAN BE APPROXIMATED BY PIECEWISE LINEAR ONES

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A homeomorphism h of E_n or S_n onto itself is stable if $\exists$ homeomorphisms $h_1, h_2, \cdots, h_m$ and nonvoid open sets $U_1, U_2, \cdots, U_m$ such that $h = h_m h_{m-1} \cdots h_1$ and $h_i | U_i = I$ for $i = 1, 2, \cdots, m$. All orientation preserving homeomorphisms of E_n or S_n are stable provided $n = 1, 2,$ or 3. There is no example known in any dimension of an orientation preserving homeomorphism which is not stable. In fact, the conjecture that all orientation preserving homeomorphisms of E_n or S_n are stable is equivalent to the annulus conjecture (see [3]).

It is known that any homeomorphism of E_3 onto itself can be approximated by a piecewise linear one (see [2] or [6]). The purpose of this paper is to announce that if $n \geqq 7$ and h is a stable homeomorphism of E_n or S_n onto itself, then h can be approximated by a piecewise linear homeomorphism, and also, in the case of E_n, by a diffeomorphism.

The set of all homeomorphisms on E_n or S_n forms a group under composition and the subset of stable homeomorphisms forms a normal subgroup. The stable group on S_n is simple while the stable group on E_n is not. Due to this fact, there is a shorter proof in the case of S_n than in the case of E_n, and it is this proof which will be outlined here. The author thanks John Stallings for his assistance.

NOTATION. E_n is Euclidean n-space, S_{n-1} is the unit sphere in E_n, and O_n is the open unit ball in E_n. Thus $O_n \cup S_{n-1} = \overline{O}_n$. For a given integer n, O_n will usually be denoted by O. If $U \subset E_n$ and $a > 0$, $aU = \{x \in E_n : \exists y \in U \text{ such that } x = ay\}$. $C(aU)$, the compliment of aU, will be denoted by Ua. Thus, for a given n, aO will be the canonical open ball in E_n of radius a. If $x, y \in E_n$, $|x - y|$ will be the usual distance from x to y. If O is the origin and $x \neq O \neq y$, then $\theta\{x, y\}$ will represent the angle in radians between the two line intervals, one joining O to x and the other joining O to y. Thus $0 \leqq \theta\{x, y\} \leqq \pi$. A piecewise linear structure (p.w.l. structure) or combinatorial structure on an open subset of E_n or S_n is a triangulation such that the star of each vertex is a combinatorial cell (see §3 of [10]). The identity function will be denoted by I.

The results of this paper are based primarily on Lemma 1 below, a modification of the Engulfing Lemma (see §3.4 of [10]). The proof is omitted.

LEMMA 1. *Suppose E_n ($n \geqq 4$) has an arbitrary p.w.l. structure T, K is a finite subcomplex of T,* dim $K \leqq n-4$, *a, b, and ϵ are nos. with $0<a<b$, $\epsilon>0$ and $K \subset bO = bO_n$. Then $\exists$ a homeomorphism $h: E_n \to E_n$ such that h is p.w.l. relative to T, $h|(a-\epsilon)O=I$, $h|Ob=I$, $h(aO) \supset K$ and $\theta\{h(x), x\} < \epsilon$ for $x \in E_n$.*

The proof of Lemma 2 below follows from Lemma 1 and trivial modifications of §4 of [10] and §8.1 of [11]. The proof is omitted.

LEMMA 2. *Suppose E_n ($n \geqq 7$) has an arbitrary p.w.l. structure T, and a, b, and ϵ are nos. with $0<a<b$ and $\epsilon>0$. Then $\exists$ a homeomorphism $h: E_n \to E_n$ such that h is p.w.l. relative to T, $h|(a-\epsilon)O=I$, $h|O(b+\epsilon) = I$, $h(aO) \supset bO$ and $\theta\{h(x), x\} < \epsilon$ for $x \in E_n$.*

DEFINITION. A homeomorphism $h: S_n \to S_n$ is said to have property P if for any p.w.l. structure T on S_n and any $\epsilon>0$, $\exists$ a homeomorphism $f: S_n \to S_n$ such that f is p.w.l. relative to T and $|h(x)-f(x)| < \epsilon$ for $x \in S_n$. Let G_n be the set of all homeomorphisms on S_n which possess property P.

OBSERVATION A. G_n is a normal subgroup of the group of all homeomorphisms under composition.

PROOF. The proof that it is a subgroup is immediate. It will be shown that G_n is normal. Suppose $h \in G_n$ and $g: S_n \to S_n$ is any homeomorphism. Show that $g^{-1}hg \in G_n$. Let T and ϵ be given.

There exists a $\delta>0$ such that if $|x-y| < \delta$, then $|g^{-1}(x)-g^{-1}(y)| < \epsilon$. Let T_1 be the p.w.l. structure on S_n which is the g image of T, $T_1 = g(T)$. Thus if v is a simplex of S_n in the triangulation T, then $g(v)$ is a simplex of S_n in the triangulation T_1. Since $h \in G_n$, $\exists$ a homeomorphism $f: S_n \to S_n$ which is p.w.l. relative to T_1 and with $|h(x)-f(x)| < \delta$ for $x \in S_n$. Thus $|g^{-1}hg(x) - g^{-1}fg(x)| < \epsilon$ for $x \in S_n$. Note that $g^{-1}fg$ is p.w.l. relative to T because: g is p.w.l. from T to T_1, f is p.w.l. from T_1 to T_1 and g^{-1} is p.w.l. from T_1 to T. This justifies Observation A.

THEOREM 1. *Let T be an arbitrary p.w.l. structure on S_n ($n \geqq 7$) and let $h: S_n \to S_n$ be a stable homeomorphism. If $\epsilon>0$, $\exists$ a homeomorphism $f: S_n \to S_n$ such that f is p.w.l. relative to T and $|h(x)-f(x)| < \epsilon$ for $x \in S_n$.*

PROOF. The set of all stable homeomorphisms of S_n is a simple, normal subgroup of the group of all homeomorphisms. The fact that it is a normal subgroup is trivial and the fact that it is simple follows from [1] and is even stated explicitly in Theorem 14 of [4]. Therefore, using Observation A, it will follow that G_n contains the stable group if G_n contains some stable homeomorphism distinct from the identity. This will now be shown.

Let h be a symmetric radial expansion, i.e., let $h: E_n \to E_n$ be a homeomorphism such that $h(x) = x$ for $\|x\| \geqq 1$, $h(0) = 0$, $\theta\{h(x), x\} = 0$ for all x, and if $0 < r < 1$, $\exists$ a no. $u(r)$, $r < u(r) < 1$ such that $h[r(\bar{O} - O)] = u(r)(\bar{O} - O)$. Let T be any p.w.l. structure on E_n and $\epsilon > 0$. It will be shown that $\exists f: E_n \to E_n$ which is a p.w.l. homeomorphism relative to T and with $f(x) = x$ for $\|x\| \geqq 1$ and $|h(x) - f(x)| < \epsilon$ for $x \in E_n$. Since h determines a homeomorphism from S_n to itself by defining $h(\infty) = \infty$, this will show that G_n is nontrivial and will complete the proof of Theorem 1.

Let $0 = r_0 < r_1 < r_2 \cdots < r_{m+1} = 1$ be nos. such that $(u(r_{i+2}) - u(r_i)) < \epsilon/2$ for $i = 0, 1, 2, \cdots, (m-1)$. By Lemma 3, $\exists$ p.w.l. homeomorphisms $f_1, f_2, \cdots, f_m$ such that $f_i | r_{i-1}O = I$, $f_i | Ou(r_{i+1}) = I$, $\theta\{f_i(x), x\} < \epsilon/4$ for $x \in E_n$, and $f_i(r_iO) \supset u(r_i)O$ for $i = 1, 2, \cdots, m$. Let $f = f_1 f_2 \cdots f_m$. Now f is a homeomorphism of E_n onto E_n that is p.w.l. relative to T and $f | C(O) = I$. It will be shown that $|f(x) - h(x)| < \epsilon$ for $x \in O$. Let $x \in r_{k+1}O \cap Or_k = r_{k+1}O - r_kO$, $0 \leqq k \leqq m$. Then $f(x) = f_1 f_2 \cdots f_{k+1}$ because $f_t | r_{k+1}O = I$ for $t > k+1$. In fact, $f(x) = f_k f_{k+1}(x)$ because $f_k f_{k+1}(x) \in Ou(r_k)$ and $f_t | Ou(r_k) = I$ for $t < k$. (In the special case $k = 0$, $f(x) = f_1(x)$.) Now since $f(x)$ and $h(x) \in u(r_{k+2})O \cap Ou(r_k)$, $\|f(x)\|$ and $\|h(x)\|$ differ by $< \epsilon/2$. Since $\theta\{h(x), f(x)\} < \epsilon/2$ is measured in radians and any radius under consideration is < 1, it follows that $|h(x) - f(x)| < \epsilon$. This completes the proof.

THEOREM 2. *Let T be an arbitrary p.w.l. structure on $E_n (n \geqq 7)$. If $h: E_n \to E_n$ is a stable homeomorphism and $\epsilon(x): E_n \to (0, \infty)$ is a continuous function, then $\exists$ a homeomorphism $f: E_n \to E_n$ such that f is p.w.l. relative to T and $|f(x) - h(x)| < \epsilon(x)$ for $x \in E_n$.*

Since the stable group on E_n is not simple, the trick used in the proof of Theorem 1 cannot be used. A direct construction of f is required. The proof is omitted.

THEOREM 3. *Suppose D is any C^2 differentiable structure on E_n $(n \geqq 7)$. If $h: E_n \to E_n$ is a stable homeomorphism and $\epsilon(x): E_n \to E_n$ is a continuous function, then $\exists$ a homeomorphism $f: E_n \to E_n$ which is a C^2 diffeomorphism relative to D and such that $|f(x) - h(x)| < \epsilon(x)$ for $x \in E_n$.*

PROOF. Let T be a C^2 triangulation of E_n which is compatible with D (see [5] or [13]). By Theorem 2, h may be approximated by a homeomorphism f_1 which is p.w.l. relative to T. Now by Theorems 5.7 and 6.2 of [7], f_1 may be approximated by a diffeomorphism f. This completes the proof. The theorem remains true if C^2 is replaced by C^∞.

It is not clear whether or not Theorem 3 remains true when E_n is replaced by S_n. It is known that E_n and S_n have to be considered as separate cases. For instance, any two differentiable structures on E_n are equivalent (except possibly for $n=4$) while this is not true on S_n (see [10] and [5] respectively).

Let D_1 and D_2 be two differentiable structures on E_n ($n \geqq 7$). Let $h: E_n \rightarrow E_n$ be a diffeomorphism mod D_1. Since diffeomorphisms are always stable, according to Theorem 3, h can be approximated by f, a diffeomorphism mod D_2. This type of question might be interesting on S_n. For instance, let $n=7$ and D_1 be the ordinary differentiable structure on S_7 and D_2 be one of Milnor's bad differentiable structures. Can diffeomorphisms mod D_1 be approximated by diffeomorphisms mod D_2? They can be approximated by p.w.l. ones by Theorem 1.

If T_1 and T_2 are two p.w.l. structures on E_n (or S_n), then any homeomorphism p.w.l. relative to T_1 can be approximated by one p.w.l. relative to T_2. This follows from Theorem 2 (resp. Theorem 1) and the fact that p.w.l. homeomorphisms are stable.

References

1. R. D. Anderson, *The algebraic simplicity of certain groups of homeomorphisms*, Amer. J. Math. **80** (1958), 955–963.

2. R. H. Bing, *An alternate proof that 3-manifolds can be triangulated*, Ann. of Math. (2) **69** (1959), 37–65.

3. M. Brown and H. Gluck, *Stable structures on manifolds*. I. *Homeomorphisms of S_n*, Ann. of Math. (to appear).

4. G. M. Fisher, *On the group of all homeomorphisms of a manifold*, Trans. Amer. Math. Soc. **97** (1960), 193–212.

5. J. Milnor, *On the relationship between differentiable manifolds and combinatorial manifolds*, mimeographed notes, Princeton University, Princeton, N. J., 1956.

6. E. E. Moise, *Affine structures in 3-manifolds*. IV. *Piecewise linear approximations of homeomorphisms*, Ann. of Math. (2) **55** (1952), 215–222.

7. J. Munkres, *Obstructions to the smoothing of piecewise-differentiable homeomorphisms*, Ann. of Math. (2) **72** (1960), 521–554.

8. R. Penrose, J. H. C. Whitehead and E. C. Zeeman, *Imbedding of manifolds in Euclidean space*, Ann. of Math. (2) **73** (1961), 613–623.

9. J. Stallings, *Polyhedral homotopy spheres*, Bull. Amer. Math. Soc. **66** (1960), 485–488.

10. ———, *The piecewise-linear structure of euclidean space*, Proc. Cambridge Philos. Soc. **58** (1962), 481–488.

11. ———, *On topologically unknotted spheres*, Ann. of Math. (to appear).

12. J. H. C. Whitehead, *Simplicial spaces, nuclei and m-groups*, Proc. London Math. Soc. **45** (1939), 243–327.

13. ———, *On C^1-complexes*, Ann. of Math. (2) **41** (1940), 809–824.

14. M. K. Fort, *Topology of 3-manifolds*, Prentice-Hall, Englewood Cliffs, N. J., 1962, pp. 198–204.

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