UNKNOTTING $S^1$ IN $S^4$

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Topologists have for some time suspected that the $k$-sphere $S^k$ can be topologically knotted in the $n$-sphere $S^n$ if and only if $k > 0$ and $n - k = 2$. Strictly speaking, this is not quite correct (because of the existence of wild embeddings), but with the appropriate local flatness condition, the conjecture has been verified by Brown [1; 2] for $n - k = 1$, Artin [3] for $n - k = 2$, and Stallings [4] for $n - k \geq 3$, the single undecided case occurring when $k = 1$ and $n = 4$.

It is the object of this note to show that, on the basis of some recent results of Homma, $S^1$ cannot be knotted in $S^4$.

1. **The main theorem.** $\mathbb{R}^n$ will denote $n$-dimensional Euclidean space, and we identify $\mathbb{R}^n$ with $\mathbb{R}^n \times \mathbb{R}^{n+1}$ so that we may write $\mathbb{R}^n \subset \mathbb{R}^{n+1}$. The unit sphere in $\mathbb{R}^{n+1}$ will be denoted by $S^n$. $S^n$ can be triangulated as a combinatorial manifold so that, for each $k < n$, $S^k$ appears as a subcomplex.

Let $f$ be an embedding of a $k$-manifold $M^k$ in an $n$-manifold $M^n$ with the property that each point of $f(M^k)$ has a neighborhood $U$ in $M^n$ such that the pair $(U, U \cap f(M^k))$ is homeomorphic to the pair $(\mathbb{R}^n, R^k)$. Then $f$ is called a **locally flat** embedding and $f(M^k)$ is called a **locally flat** submanifold of $M^n$.

The main theorem of this paper will be

**Theorem 1.1.** Let $f_1$ and $f_2$ be locally flat embeddings of $S^1$ in $S^4$. Then there is a homeomorphism $h$ of $S^4$ onto itself such that

$$hf_1 = f_2.$$ 

Furthermore, if $p$ is a point of $S^4 - f_1(S^1) - f_2(S^1)$, then $h$ can be chosen so as to restrict to the identity in some neighborhood of $p$.

Since a general position argument will prove Theorem 1.1 whenever $f_1$ and $f_2$ happen to be piecewise linear embeddings, it will be more than sufficient to prove the following theorem, in which $U_\epsilon(f(S^1))$ denotes the set of points in $S^4$ whose distance from $f(S^1)$ is less than $\epsilon$.

**Theorem 1.2.** Let $f$ be a locally flat embedding of $S^1$ in $S^4$. Then for any $\epsilon > 0$, there is an $\epsilon$-homeomorphism $h$ of $S^4$ onto itself such that

$$h/S^4 - U_\epsilon(f(S^1)) = 1,$$

$$hf : S^1 \to S^4$$ is piecewise linear.
2. Homma's results. Homma [5] has recently proved the following theorem.

**Homma's Theorem.** Let the following be given:

(i) $\mathcal{M}^n$, a finite combinatorial $n$-manifold;

(ii) $\mathcal{M}^n$, a finite combinatorial $n$-manifold topologically embedded in $\mathcal{M}^n$;

(iii) $\mathcal{P}^k$, a finite polyhedron piecewise linearly embedded in $\text{int}(\mathcal{M}^n)$.

If $2k+2 \leq n$, then for any $\epsilon > 0$ there is an $\epsilon$-homeomorphism $F$ of $\mathcal{M}^n$ onto $\mathcal{M}^n$ such that

$$F/\mathcal{M}^n - U_\epsilon(\mathcal{P}^k) = 1,$$

$$F/\mathcal{P}^k$$

is piecewise linear.

With only slight modifications, Homma's arguments are sufficient to produce the following somewhat more general result.

**Theorem 2.1.** Let the following be given:

(i) $\mathcal{M}^n$, a possibly noncompact combinatorial $n$-manifold;

(ii) $\mathcal{M}^n$, a possibly noncompact combinatorial $n$-manifold, topologically embedded in $\mathcal{M}^n$;

(iii) $\mathcal{P}^k$, a possibly infinite polyhedron, piecewise linearly embedded as a closed subset of $\text{int}(\mathcal{M}^n)$;

(iv) $\mathcal{L}$, a subpolyhedron of $\mathcal{P}^k$ such that $\text{Cl}(\mathcal{P}^k - \mathcal{L})$ is a finite polyhedron, and such that $\mathcal{L}$ is piecewise linearly embedded in $\mathcal{M}^n$ as well as $\mathcal{M}^n$.

If $2k+2 \leq n$, then for any $\epsilon > 0$ there is an $\epsilon$-homeomorphism $F$ of $\mathcal{M}^n$ onto $\mathcal{M}^n$ such that

$$F/\mathcal{M}^n - U_\epsilon(\mathcal{P}^k - \mathcal{L}) = 1,$$

$$F/\mathcal{L} = 1,$$

$$F/\mathcal{P}^k$$

is piecewise linear.

3. Proof of the main theorem

**Lemma 3.1.** Let $\alpha$ be an open arc in $S^4$, and $u,v,w,x$ four points on $\alpha$, in that order. Let $U$ and $V$ be open neighborhoods in $S^4$ of the closed subarcs $[uw]$ and $[vx]$, respectively, of $\alpha$, such that $(U, U \cap \alpha) \approx (R^4, R^3)$ $\approx (V, V \cap \alpha)$. Then there is an open neighborhood $W$ of $[ux]$ in $S^4$ such that $(W, W \cap \alpha) \approx (R^4, R^3)$.

Since $(U, U \cap \alpha) \approx (R^4, R^3)$, there is a homeomorphism $h$ of $U$ onto itself which takes $U \cap \alpha$ onto itself, $u$ onto $v$ and $w$ onto itself, and is the identity near the boundary of $U$. Extend $h$ over $S^4$ via the identity, and let $W = h^{-1}(V)$.

Repeated use of this lemma proves the following
Theorem 3.2. Let $S$ be a locally flat 1-sphere in $S^4$. Then $S$ may be written as the union of two open arcs, $A$ and $B$, which have neighborhoods, $U_A$ and $U_B$, in $S^4$ such that

(i) $U_A \cap S = A$ and $(U_A, A) \approx (R^4, R^1)$;
(ii) $U_B \cap S = B$ and $(U_B, B) \approx (R^4, R^1)$.

Now let $f$ be a locally flat embedding of $S^1$ in $S^4$, and $\epsilon > 0$ a given positive number. Theorem 1.2 will be proved by a double application of Homma’s theorem, first in its original form and then in the form of Theorem 2.1.

Proof of Theorem 1.2. Write $f(S^1)$ as the union of two open arcs $A$ and $B$ as in the above theorem, and let $x$ and $y$ be two points of $f(S^1)$, one chosen from each of the two components of $A \cap B$. Then $f(S^1)$ is the union of the two closed arcs $a \subset A$ and $b \subset B$, which intersect at $x$ and $y$.

Step 1. Since $(U_A, A) \approx (R^4, R^1)$, $U_A$ can be triangulated as a combinatorial manifold in such a way as to make

$$f: f^{-1}(a) \to a \subset U_A$$

a piecewise linear embedding.

Let $M^n = S^4$, $\bar{M}^n = a$ a closed regular neighborhood of $a$ in $U_A$, and $\bar{P}^k = a$. Homma’s theorem then asserts the existence of an $\epsilon/2$-homeomorphism $F_1$ of $S^4$ onto itself such that

$$F_1/S^4 - U_{\epsilon/2}(a) = 1,$$
$$F_1/a$$

is piecewise linear.

Step 2. Since $(F_1(U_B), F_1(B)) \approx (U_B, B) \approx (R^4, R^1)$, $F_1(U_B)$ can be triangulated as a combinatorial manifold in such a way as to make

$$F_1f: f^{-1}(B) \to F_1(B) \subset F_1(U_B)$$

a piecewise linear embedding.

For the second application of Homma’s theorem, let $M^n = F_1(U_B)$ triangulated as an open subset of $S^4$, $\bar{M}^n = F_1(U_B)$ triangulated as in the preceding paragraph, $\bar{P}^k = F_1(B)$ and $\bar{L} = F_1(B) \cap F_1(a)$. Note that by choice of $F_1$, $\bar{L}$ is piecewise linearly embedded in $M^n$ as well as in $\bar{M}^n$. Now apply Theorem 2.1 to obtain an $\epsilon/2$-homeomorphism $F_2$ of $F_1(U_B)$ onto itself such that

$$F_2/F_1(U_B) - U_{\epsilon/2}(F_1(B) - F_1(a)) = 1,$$
$$F_2/F_1(B) \cap F_1(a) = 1,$$
$$F_2/F_1(B)$$

is piecewise linear.

$F_2$, which is the identity near the boundary of $F_1(U_B)$, may be ex-
tended via the identity to a homeomorphism $F_2$ of $S^4$ onto itself.

Then $h = F_2F_1$ is an $\epsilon$-homeomorphism of $S^4$ onto itself such that

$$h/S^4 = U_\epsilon(f(S^4)) = 1,$$

$h_f: S^1 \to S^3$ is piecewise linear.

This completes the proof of Theorem 1.2, and hence also of Theorem 1.1.

Theorem 1.2 is actually a very special case of a more general result which will be described elsewhere.

**References**


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