EXISTENCE THEOREM FOR THE BARGAINING SET $M_i^{(g)}$

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M. Davis and M. Maschler have conjectured [1] that for each coalition structure $B$ in a cooperative game, there exists a payoff vector $x$ such that the payoff configuration $(x; B)$ is stable, i.e., belongs to the bargaining set $M_i^{(g)}$. We outline here a proof of the conjecture. The details of the proof will be published elsewhere.

Let $B = B_1, B_2, \ldots, B_m$ be a fixed coalition structure for an $n$-person game $\Gamma$ with a characteristic function $v(B)$, satisfying $v(B) \geq 0$, and $v(i) = 0$ for $i = 1, 2, \ldots, n$. We denote by $X(B)$ the space of the points $x = (x_1, x_2, \ldots, x_n)$ such that $(x; B)$ is an individually rational payoff configuration (i.r.p.c.). Thus, $X(B) = S_1 \times S_2 \times \cdots \times S_m$, where for $j = 1, 2, \ldots, m$, $S_j$ is the simplex $\{x_j\}_{k \in B_j}$.

LEMMA. Let $c^1(x), c^2(x), \ldots, c^n(x)$ be non-negative continuous real functions defined for $x \in X(B)$. If, for each $x$ in $X(B)$, and for each coalition $B_j$ in $B$, there exists a player $i$ in $B_j$, such that $c^i(x) \geq x_i$, then there exists a point $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$ in $X(B)$ such that $c^k(\xi) \geq \xi_k$ for all $k, k = 1, 2, \ldots, n$.

The proof is indirect and one arrives at the contradiction by using Brouwer’s fixed point theorem.

Let $(x; B)$ be an i.r.p.c. We shall denote by $(\hat{y}^{B_1}, \hat{y}^{N-B_1}; B)$ an i.r.p.c. which results from the previous one by holding the payments to the players in $N - B_1$ fixed, and giving each player $k$ in $B_j, B_j \subset B$, an amount $y_k$. Clearly, $x^{N-B_1}$ is the projection of $x$ on the space $S_i \times \cdots \times S_{j-1} \times S_{j+1} \times \cdots \times S_m$, and $\hat{y}^{B_1} = \{y_k\}$ is a point in $S_j$.

Let $E_j(x)$ be the set of points $\hat{y}^{B_1}$ in $S_j$, having the property that in $(\hat{y}^{B_1}, \hat{y}^{N-B_1}; B)$, player $i, i \in B_j$, is not weaker than any other player. The set $E_j(x)$ is closed and contains the face $y_i = 0$ of $S_j$. (See [2].)

We now define for each player $i$, $i = 1, 2, \ldots, n$, the function

$$c^i(x) = x_i + \max_{\hat{y}^{B_1} \in E_j(x)} \min_{k \in B_1} (x_k - y_k).$$

Here, $B_j$ is that coalition of $B$ which contains player $i$.

1 Throughout this paper we shall use the definitions and the notations of [2].
2 Another proof has been given by the author, M. Davis, and M. Maschler. It has been decided to publish this version, which is simpler.
It can be shown that \( c^i(\xi) \) is a non-negative continuous function of \( \xi \).

Since \( \sum_{k \in B_i} x_k = \sum_{k \in B_i} y_k = v(B_i) \), it follows that \( c^i(x) \leq x_i \) for all \( i, i=1, 2, \ldots, n \). Let \( E_i, i=1, 2, \ldots, n \), be the set of points \( x, x \in X(B) \), for which \( i \) is not weaker than any other player of the coalition \( B_j \) which contains player \( i \). Clearly, \( (x; B) \in \mathcal{M}_i^{(0)} \) if and only if \( x \in \bigcap_{k=1}^n E_k \). If \( x \in E_i \), then its projection \( \hat{x}^{B_i} \) on \( S_i \) belongs to \( E_i(x) \). In this case \( c^i(x) = x_i \). Conversely, if \( c^i(x) = x_i \), then some \( \bar{x}^{B_i} \in E_i(x) \) must be equal coordinatewise to \( x^{B_i} \), hence \( \xi \in E_i \).

It is proved in [2] (see proof of Theorem 2), that for each \( x, x \in X(B) \), and for each coalition \( B_j, B_j \in B \), there exists a player \( i, i \in B_j \), such that \( x \in E_i \). Thus, for this player, \( c^i(x) = x_i \). By the lemma, there exists a point \( \xi, \xi \in X(B) \), such that \( c^k(\xi) = \xi_k \) for all \( k, k=1, 2, \ldots, n \). Therefore, \( \xi \in \bigcap_{k=1}^n E_k \), and so \( (\xi, B) \in \mathcal{M}_i^{(0)} \). We have thus proved:

**Theorem.** Let \( B \) be a coalition structure in an \( n \)-person cooperative game; then there always exists a payoff vector \( x \) such that \( (x; B) \in \mathcal{M}_i^{(0)} \).

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**References**


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