

A POLYNOMIAL ANALOG OF THE GOLDBACH CONJECTURE¹

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We call a polynomial $c_0x^m + c_1x^{m-1} + \dots + c_m$ in the ring $GF[q, x]$ *primary* if $c_0=1$. Suppose H is a polynomial in $GF[q, x]$ and let $h = \deg H$. Then the following theorem is easily established:

THEOREM 1. *If q is sufficiently large relative to h , then H is the sum of two irreducible polynomials, each of degree $h+1$.*

PROOF. The primary irreducibles of degree $h+1$ fall into $\phi(H)$ residue classes mod H . The number of such irreducibles is

$$\frac{q^{h+1}}{h+1} + O\left(\frac{q^{(h+1)/2}}{h+1}\right)$$

and the number of residue classes is $\phi(H) < q^h$. Therefore, if q is sufficiently large relative to h , some one residue class contains two irreducibles P and Q . For any such pair of irreducibles P and Q , there is an element α of $GF(q)$ so that $\alpha P + (-\alpha)Q = H$. This is the assertion of the theorem.

Our aim in this note is to sketch a proof of an asymptotic formula ($q \rightarrow \infty$) for the number of representations of the polynomial H as a sum of two irreducibles, each of degree $h+1$. More specifically,

THEOREM 2. *Let $N(H)$ denote the number of pairs P, Q of primary irreducibles in $GF[q, x]$ such that*

- (1) $\deg P = \deg Q = h+1$,
- (2) $P \neq Q$,
- (3) $P - Q \equiv 0 \pmod{H}$.

Then we have the asymptotic formula

$$(1) \quad N(H) = \frac{q^{2(h+1)}}{(h+1)^2 \phi(H)} + O(q^{h+1}) \quad \text{as } q \rightarrow \infty.$$

OUTLINE OF PROOF. Let $\pi(r; H, K)$ denote the number of primary irreducibles P of degree r such that $P \equiv K \pmod{H}$. Then we have

$$(2) \quad N(H) = \sum_K [\pi(h+1; H, K)]^2 - \psi(h+1),$$

where K runs through a reduced residue system mod H , and $\psi(r)$ is

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the number of primary irreducibles in $GF[q, x]$ of degree r .

Let $\pi_K(r, d)$ denote the number of primary irreducibles P in $GF[q, x]$ such that

- (1) $\deg P = r/d,$
- (2) $P^d \equiv K \pmod{H}$

and let $D(r, K) = \sum_{d|r} (1/d)\pi_K(r, d)$. Note that $\pi_K(r, 1) = \pi(r; H, K)$. If $r < 2h$, we have $\pi_K(r, d) \leq d$ for $d > 1$. Thus, we find that

$$(3) \quad D(r, K) - \pi(r; K, H) \leq r,$$

for $r < 2h$.

According to a formula of Artin [1], we have

$$(4) \quad r\phi(H)D(r, K) = q^r - \sum_{\chi} \bar{\chi}(K) \sum_{i=1}^{m(\chi)} \beta_i^r(\chi)$$

where χ runs through all characters mod H and $m(\chi) \leq h$. The numbers $\beta_i(\chi)$ for $1 \leq i \leq m(\chi)$ are closely associated with the zeroes of the L -function $L(s, \chi)$. The association is such that it follows from the Riemann hypothesis for algebraic function fields [2] that

$$(5) \quad |\beta_i(\chi)| \leq hq^{1/2}$$

for all χ and $1 \leq i \leq m(\chi)$.

Using (3) and (4), we can show that

$$(6) \quad N(H) = \sum_K [D(h+1, K)]^2 - \psi(h+1) + O(q^{h+1}).$$

This last formula does not require (5). Also, after some manipulation, we derive from (4) and (5) the formula

$$(7) \quad \sum_K [D(h+1, K)]^2 = \frac{q^{2(h+1)}}{(h+1)^2\phi(H)} + O(q^{h+1}).$$

Combining (6) and (7) and making use of the trivial estimate $\psi(r) = O(q^r)$, we arrive at (1). This completes the proof.

REFERENCES

1. E. Artin, *Quadratische Körper im Gebiete der höheren Kongruenzen*. II, Math. Z. **19** (1924), 242-246.
2. A. Weil, *Sur les courbes algébriques et les variétés qui s'en déduisent*, Actualités Sci. Ind. No. 1041, Hermann, Paris, 1945, Deuxième Partie, §V.

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