# RESEARCH ANNOUNCEMENTS

The purpose of this department is to provide early announcement of significant new results, with some indications of proof. Although ordinarily a research announcement should be a brief summary of a paper to be published in full elsewhere, papers giving complete proofs of results of exceptional interest are also solicited.

## THE MAXIMAL SEMILATTICE DECOMPOSITION OF A SEMIGROUP

BY MARIO PETRICH

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1. The maximal semilattice homomorphic image of a semigroup $S$ is the semilattice (commutative idempotent semigroup) $Y$ such that every semilattice homomorphic image of $S$ is also a homomorphic image of $Y$. The maximal semilattice decomposition of $S$ is the decomposition of $S$ into equivalence classes which are complete inverse images of members of $Y$. We identify these classes with members of $Y$.

$S$ will denote any semigroup and $x$ any element of $S$ unless stated otherwise. We follow the notation and terminology of [2]. Proofs of statements in this note will appear elsewhere. A subsemigroup $N$ of $S$ is called a face of $S$ if, for all $x, y \in S$, $xy \in N$ implies $x, y \in N$. A subset $N$ of $S$ is a face of $S$ if and only if its complement in $S$ is a prime ideal of $S$ or is empty.

DEFINITION. Let $N(x)$ be the smallest face of $S$ containing $x$ and $N_x = \{y \in S \mid N(x) = N(y)\}$. The sets $N_x$ will be called $N$-classes and $Y$ will denote the set of all distinct $N$-classes of $S$ together with the operation $N_x N_y = N_{xy}$.

$N(x)$ is the intersection of all faces of $S$ containing $x$, $N_x$ is a subsemigroup of $S$, $N$-classes define an equivalence relation on $S$, and $Y$ is a semilattice. Theorems 1 and 3 are our fundamental results.

THEOREM 1 (cf. [1; 6]). *$Y$ is the maximal semilattice decomposition of $S$.*

PROOF. Let $Z$ be any semilattice decomposition of $S$ ($Z = S$ is such a decomposition). Let $B_x$ denote the member of $Z$ containing $x$. Let $x, y \in S$ and suppose that $y \notin B_x$. Then $B_x \neq B_y$ and thus either $B_x < B_y$ or $B_x \nleq B_y$. In the first case we let $T = \bigcup_{B_z \geq B_y} B_z$ and in the second $T = \bigcup_{B_z \geq B_x} B_z$. It is clear that in either case $T$ is a face of

$S$ and that in the first case $x \notin T$, $y \in T$ and in the second $x \in T$, $y \notin T$. Therefore $y \notin N_x$ and thus $N_x \subseteq B_x$.

The sets $N(x)$ and $N_x$ have a number of interesting properties.

THEOREM 2. *Let $N_1(x) = \langle x \rangle$, for $n \geqq 1$ let $N_{n+1}(x)$ be the semigroup generated by all elements $y$ of $S$ such that $N_n(x) \cap J(y) \neq \square$. Then $N(x) = \bigcup_{n=1}^{\infty} N_n(x)$. Moreover $N(x) = \bigcup_{N_y \geqq N_x} N_y$.*

THEOREM 3 (cf. [5]). *No ideal of any $N$-class contains prime ideals.*

COROLLARY 1. *$S$ contains no prime ideals if and only if $S$ is a single $N$-class.*

COROLLARY 2. *There exists a one-to-one isotone (with respect to inclusion) mapping of the set of all prime ideals of $S$ onto the set of all prime ideals of $Y$.*

THEOREM 4. *$N_x$ is the largest subsemigroup of $S$ containing $x$ and containing no prime ideals. Moreover, $N_x = \{y \in S \mid \chi(x^n) = \chi(uy)$ and $\chi(y^n) = \chi(vx)$ for some $u, v \in S$, some natural number $n$, and all semicharacters $\chi$ of $S\}$.*

2. Some interesting connections between the properties of each $N$-class and the whole semigroup $S$ can be established; in particular properties concerning elements or ideals of $N$-classes and $S$. As an example we state two theorems.

THEOREM 5. *These are equivalent:*

(a) *every $N$-class is a group;*

(b) *every left and every right ideal of every $N$-class is semiprime and two-sided;*

(c) *every left and every right ideal of $S$ is semiprime and two-sided;*

(d) *for every $x \in S$, $x \in Sx^2 \cap x^2S$ and $xS = Sx$;*

(e) *for every $x \in S$, $N_x = H_x$.*

*Moreover, if any of these conditions holds, then $N_x = \{y \in S \mid xS = yS\}$.*

THEOREM 6. *These are equivalent:*

(a) *$Y$ is linearly ordered;*

(b) *the set of prime ideals is linearly ordered under inclusion;*

(c) *every nonempty intersection of prime ideals is a prime ideal.*

3. We give explicit expressions for the sets $N(x)$ or $N_x$ for certain classes of semigroups.

THEOREM 7. *For every $x \in S$, $SxS = Sx^2S$ if and only if, for every $x \in S$, $N(x) = \{y \in S \mid SxS \subseteq SyS\}$.*

COROLLARY (cf. [3]). *If $S$ is a band, then for every $x \in S$, $N(x) = \{y \in S \mid x = xyx\}$.*

DEFINITION. $S$ is said to be weakly commutative if, for any $x, y \in S$, $(xy)^k = ax = yb$ for some $a, b \in S$, and some natural number $k$.

THEOREM 8 (cf. [4]). *If $S$ is weakly commutative, then $N(x) = \{y \in S \mid \langle x \rangle \cap Sy \neq \square\}$.*

COROLLARY (cf. [5]). *If $S$ is weakly commutative, then these are equivalent:*

(a) *$S$ contains no proper semiprime [left] ideals;*

(b) *$S$ contains no prime [left] ideals;*

(c) *for every $x, y \in S$, $\langle x \rangle \cap Sy \neq \square$.*

*If $S$ is periodic, let $E$ be the set of all idempotents of $S$, and for $e \in E$, let $K^{(e)} = \{x \in S \mid x^n = e$ for some natural number $n\}$. $S$ is strongly reversible if, for any $x, y \in S$, $(xy)^r = x^s y^t = y^t x^s$ for some $r, s, t$.*

THEOREM 9. *If $S$ is periodic, then for every $x \in S$, $N_x = K^{(e)}$, where $x \in K^{(e)}$, if and only if $S$ is weakly commutative.*

COROLLARY (cf. [7]). *If $S$ is periodic, then these are equivalent:*

(a) *$K^{(e)}K^{(f)} \subseteq K^{(ef)} = K^{(fe)}$ for all $e, f \in E$;*

(b) *$S$ is strongly reversible;*

(c) *$S$ is weakly commutative and $E$ is a semigroup.*

*Moreover, if any of these conditions holds, then $N(x) = \bigcup_{f \geq e} K^{(f)}$ where $x \in K^{(e)}$, and $Y$ and $E$ are isomorphic.*

## REFERENCES

1. A. H. Clifford, Review of [6], Math. Reviews **17** (1956), 584.

2. A. H. Clifford and G. B. Preston, *The algebraic theory of semigroups*, Vol. 1, Math. Surveys No. 7, Amer. Math. Soc., Providence, R.I., 1961.

3. D. McLean, *Idempotent semigroups*, Amer. Math. Monthly **61** (1954), 110–113.

4. T. Tamura and N. Kimura, *On decompositions of a commutative semigroup*, Kōdai Math. Sem. Rep. **4** (1954), 109–112.

5. G. Thierrin, *Sur quelques propriétés de certaines classes de demi-groupes*, C. R. Acad. Sci. Paris **239** (1954), 1335–1337.

6. M. Yamada, *On the greatest semilattice decomposition of a semigroup*, Kōdai Math. Sem. Rep. **7** (1955), 59–62.

7. ———, *A remark on periodic semigroups*, Sci. Rep. Shimane Univ. **9** (1959), 1–5.

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