POLYHEDRAL NEIGHBORHOODS IN
TRIANGULATED MANIFOLDS

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This note is an outline of some of the author's recent work concerning triangulated manifolds. A combinatorial structure is never assumed; indeed with this condition added, our results are mostly corollaries to well known theorems. The purpose of these investigations is two-fold: it is possible that they may lead to a proof of the cellularity of vertex stars in manifolds (a result that would have critical implications for the theory); but also, should a noncombinatorial triangulation of a manifold be found, they might serve as a starting point for the local study of such examples.

Our tools are:

I. The generalized Schoenflies theorem of Brown and Mazur \([2; 3]\).

II. Let \(M\) be a compact Hausdorff space which is the union of two open sets each of which is a homeomorph of \(E^n\); then \(M\) is homeomorphic to \(S^n\) (we write \(M \approx S^n\)). This is an immediate consequence of I.

III. If the cone over \(Y (= C(Y))\) is \(n\)-euclidean at the vertex, then the suspension of \(Y (= S(Y))\) is topologically \(S^n\). This proposition of Mazur \([3]\) follows from II.

The join of spaces \(X\) and \(Y\) is written \(X \circ Y\). The \(k\)th barycentric subdivision of a polyhedron \(P\) is denoted by \(kP\). Let \((K, L)\) be a polyhedral pair. The stellar neighborhood of \(L\) in \(K\) \((= N(K, L))\) is the union of all open simplexes of \(K\) with vertices in \(L\). The closure of \(N(K, L)\) is represented by \(St(K, L)\) (read star in \(K\) of \(L\)). For a simplex \(w\) in \(K\) let \(Lk(K, w)\) be the link of \(w\) in \(K\), and \(Cl(K, w)\) \((= w \circ Lk(K, w))\) be the cluster of \(w\) in \(K\). For a simplex \(w = u \circ v\) let \(D\) be the set of midpoints of segments from \(u\) to \(v\) and let \(B(w, u)\) be the union of all straight segments \(x \circ p\) in \(w\) with \(x \in u\) and \(p \in D\). If \(L\) is full in \(K\) define the barrel neighborhood \(B(K, L)\) of \(L\) in \(K\) as the union of all sets \(B(w, u)\) with \(w\) and \(u\) simplexes of \(St(K, L)\) and \(L\), respectively.

If \(K\) is homogeneous (in the sense of \([1]\)) then the double of \(K\), or \(2K\), consists of \(K\) and a disjoint copy \(K'\) with their combinatorial boundaries canonically identified. A quotient space of \(X\) whose only possible nondegenerate element is \(Y\) will be written \(X/Y\). A subset \(A\) of an \(n\)-manifold is cellular if it is the intersection of \(n\)-cells \((C_i)\).

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with $C_{i+1} \subseteq \text{Int } C_i$ for all $i$. If $S(X) \approx S^n$, $X$ will be called an $(n-1)$-pseudosphere.

$M$ will hereafter always stand for a triangulated $n$-manifold and $(K, L)$, for a polyhedral pair.

In [4] we have established:

**Theorem A.** Let $K$ be a full finite subpolyhedron of $M$. Then $K$ is cellular if and only if $N(M, K) \approx E^n$.

**Theorem B.** Each simplex of $^1M$ is cellular.

**Theorem C** (Added in proof). Each cluster of $^2M$ is cellular.

Next we shall sketch the proofs of:

**Theorem 1.** Let $e$ be a 1-simplex of $M$. Then $2St(^1M, ^1e) \approx S^n$.

**Corollary.** $St(^1M, ^1e) \times I \approx I^{n+1}$.

**Theorem 2.** Let $T$ be a polyhedral tree in $M$. Then $2St(^2M, ^2T) \approx S^n$.

**Corollary.** $St(^2M, ^2T) \times I \approx I^{n+1}$.

**Lemma 1.** Suppose $L$ is full in $K$. Then $N(K, L)$ is an open mapping cylinder from $\text{Bd } B(K, L)$ over $L$.

**Corollary.** $N(K, L)/L \approx C(\text{Bd } B(K, L)) - \text{Bd } B(K, L)$.

**Lemma 2.** Let $L$ be full in $K$. Then $B(K, L)$ is piecewise linearly equivalent to $St(^1K, ^1L)$.

This may be verified on each maximal simplex of $St(K, L)$ and then extended to the entire set.

**Lemma 3.** Let $K$ be a full subpolyhedron of $M$. Then $2St(^1M, ^1K)$ is an $n$-manifold.

**Lemma 4.** Suppose $e$ is a 1-simplex in $K$. Then $B = B(\text{Cl}(K, e), e) \approx C(\text{Lk}(K, e)) \times e$. If $x \in e$ then $C(\text{Lk}(K, e)) \times x = (x \circ \text{Lk}(K, e)) \cap B$.

**Lemma 5.** Let $S(X) \approx S^n$. Then $X \times I$ contains a bicollared topological $(n-1)$-sphere which separates $X_0$ from $X_1$.

**Lemma 6** (Schoenflies Theorem for Pseudospheres). Let $S(X) \approx S^n$ and $h: X \times I \rightarrow S^n$ be an imbedding. There is a homeomorphism of the pair $(S^n, h(X \times 1/2))$ onto $(S(X), X)$.

**Lemma 7.** Let $(A, B)$ be a closed pair in $S^n$ so that $A - B \approx X \times [0, 1)$, where $X$ is an $(n-1)$-pseudosphere. Then $B$ is cellular.

**Lemma 8.** Let $w$ be a $k$-simplex of $M$. Then $S^k \circ \text{Lk}(M, w) \approx S^n$. 

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Corollary. \( \text{Cl}(M, w) \times I, \, C(\text{Cl}(M, w)) \) and \( S(\text{Cl}(M, w)) \) are all homeomorphic to \( I^{n+1} \).

Proof of Theorem 1. Let \( B = B(M, e), \, e = ab \) and \( p \) be the barycenter of \( e \). For each \( x \in e \) let \( D_x = [x \circ \text{Lk}(M, e)] \cap B \). \( D_p \) divides \( B \) into two clusters \( C_a \) and \( C_b \) which are incident on \( \text{D}_p; \, C_a \) and \( C_b \) are piecewise linearly equivalent to \( \text{Cl}(M, a) \) and \( \text{Cl}(M, b) \), respectively.

Now let \( U = C_a \cup C'_a - 2D_p \) in \( 2B \). Clearly we also have \( U \approx 2C_a - D_p \subseteq 2C_a \approx S^n \). By Lemma 4 for each \( x \in ap - p, \, 2D_x \approx S(\text{Lk}(M, e)) \); the latter is an \( (n-1) \)-pseudosphere by Lemma 8. It follows by Lemmas 4 and 7 that \( U \approx E^n \). Since again by Lemma 4 \( 2D_p = \text{Bd} \, U \) is bicollared in \( 2B \), \( U \) can be expanded to an open \( n \)-cell containing \( C_a \cup C'_a \). Proposition II is now invoked to show us that \( 2B \approx S^n \).

Lemma 9. Let \( e = ab \) be a 1-simplex in \( K \) and \( p \) be the midpoint of \( e \). There is a homeomorphism of \( B(\text{Cl}(K, e), a) \) onto the barrel neighborhood of \( 1(ap) \) in \( 1[ap \circ \text{Lk}(K, e)] \); furthermore the map is the identity except possibly where it is defined in \( \text{Cl}(1K, ap) \).

This map may be found by central projection through \( a \) in each maximal simplex of \( \text{Cl}(K, e) \).

Proof of Theorem 2. This proceeds by induction on the number of vertices of \( T \). It is obvious for one vertex by III.

Assume \( T_1 \) and \( T_2 \) are disjoint nonempty trees in \( T \) and \( e \) is an edge such that \( T_1 \cup e \cup T_2 = T \). Let \( B, \, B_1 \) and \( B_2 \) be the barrel neighborhoods of \( T, \, 1T_1 \) and \( 1T_2 \), respectively, in \( 1M \). By Lemma 9 the disjoint sets \( B_1 \) and \( B_2 \) can be stretched by homeomorphisms \( h_1 \) and \( h_2 \) so that \( B = h_1(B_1) \cup h_2(B_2) \). Further \( D = h_1(B_1) \cap h_2(B_2) \subseteq p \circ \text{Lk}(M, e) \) where \( p \) is the midpoint of \( e \). It may now be seen from examining the map described in Lemma 9 that \( D \) is cellular in both \( 2h_1(B_1) \) and \( 2h_2(B_2) \); or one can deduce this from Lemma 4. The rest of the proof resembles that of Theorem 1.

Bibliography


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