POLYHEDRAL NEIGHBORHOODS IN TRIANGULATED MANIFOLDS

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This note is an outline of some of the author's recent work concerning triangulated manifolds. A combinatorial structure is never assumed; indeed with this condition added, our results are mostly corollaries to well known theorems. The purpose of these investigations is two-fold: it is possible that they may lead to a proof of the cellularity of vertex stars in manifolds (a result that would have critical implications for the theory); but also, should a noncombinatorial triangulation of a manifold be found, they might serve as a starting point for the local study of such examples.

Our tools are:

I. The generalized Schoenflies theorem of Brown and Mazur [**2**; **3**].

II. Let M be a compact Hausdorff space which is the union of two open sets each of which is a homeomorph of E^n; then M is homeomorphic to S^n (we write $M \approx S^n$). This is an immediate consequence of I.

III. If the cone over Y $(=C(Y))$ is n-euclidean at the vertex, then the suspension of Y $(=S(Y))$ is topologically S^n. This proposition of Mazur [**3**] follows from II.

The join of spaces X and Y is written $X \circ Y$. The kth barycentric subdivision of a polyhedron P is denoted by ${}^kP$. Let (K, L) be a polyhedral pair. The *stellar neighborhood of L in K* $(=N(K, L))$ is the union of all open simplexes of K with vertices in L. The closure of $N(K, L)$ is represented by $\mathrm{St}(K, L)$ (read star in K of L). For a simplex w in K let $\mathrm{Lk}(K, w)$ be the link of w in K, and $\mathrm{Cl}(K, w)$ $(=w \circ \mathrm{Lk}(K, w))$ be the cluster of w in K. For a simplex $w = u \circ v$ let D be the set of midpoints of segments from u to v and let $B(w, u)$ be the union of all straight segments $x \circ p$ in w with $x \in u$ and $p \in D$. If L is full in K define the *barrel neighborhood* $B(K, L)$ *of L in K* as the union of all sets $B(w, u)$ with w and u simplexes of $\mathrm{St}(K, L)$ and L, respectively.

If K is homogeneous (in the sense of [**1**]) then the double of K, or $2K$, consists of K and a disjoint copy K' with their combinatorial boundaries canonically identified. A quotient space of X whose only possible nondegenerate element is Y will be written X/Y. A subset A of an n-manifold is *cellular* if it is the intersection of n-cells (C_i)

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with $C_{i+1} \subseteq \operatorname{Int} C_i$ for all i. If $S(X) \approx S^n$, X will be called an $(n-1)$-*pseudosphere.*

M will hereafter always stand for a triangulated n-manifold and (K, L), for a polyhedral pair.

In [4] we have established:

THEOREM A. *Let K be a full finite subpolyhedron of M. Then K is cellular if and only if $N(M, K) \approx E^n$.*

THEOREM B. *Each simplex of ${}^1M$ is cellular.*

THEOREM C (*Added in proof*). *Each cluster of ${}^2M$ is cellular.*

Next we shall sketch the proofs of:

THEOREM 1. *Let e be a 1-simplex of M. Then* $2\operatorname{St}({}^1M, {}^1e) \approx S^n$.

COROLLARY. $\operatorname{St}({}^1M, {}^1e) \times I \approx I^{n+1}$.

THEOREM 2. *Let T be a polyhedral tree in M. Then* $2\operatorname{St}({}^2M, {}^2T) \approx S^n$.

COROLLARY. $\operatorname{St}({}^2M, {}^2T) \times I \approx I^{n+1}$.

LEMMA 1. *Suppose L is full in K. Then $N(K, L)$ is an open mapping cylinder from* Bd $B(K, L)$ *over L.*

COROLLARY. $N(K, L)/L \approx C(\operatorname{Bd} B(K, L)) - \operatorname{Bd} B(K, L)$.

LEMMA 2. *Let L be full in K. Then $B(K, L)$ is piecewise linearly equivalent to* $\operatorname{St}({}^1K, {}^1L)$.

This may be verified on each maximal simplex of $\operatorname{St}(K, L)$ and then extended to the entire set.

LEMMA 3. *Let K be a full subpolyhedron of M. Then* $2\operatorname{St}({}^1M, {}^1K)$ *is an n-manifold.*

LEMMA 4. *Suppose e is a 1-simplex in K. Then* $B = B(\operatorname{Cl}(K, e), e) \approx C(\operatorname{Lk}(K, e)) \times e$. *If $x \in e$ then* $C(\operatorname{Lk}(K, e)) \times x = (x \circ \operatorname{Lk}(K, e)) \cap B$.

LEMMA 5. *Let $S(X) \approx S^n$. Then $X \times I$ contains a bicollared topological $(n-1)$-sphere which separates X_0 from X_1.*

LEMMA 6 (SCHOENFLIES THEOREM FOR PSEUDOSPHERES). *Let $S(X) \approx S^n$ and $h: X \times I \rightarrow S^n$ be an imbedding. There is a homeomorphism of the pair $(S^n, h(X \times 1/2))$ onto $(S(X), X)$.*

LEMMA 7. *Let (A, B) be a closed pair in S^n so that $A - B \approx X \times [0, 1)$, where X is an $(n-1)$-pseudosphere. Then B is cellular.*

LEMMA 8. *Let w be a k-simplex of M. Then $S^k \circ \operatorname{Lk}(M, w) \approx S^n$.*

COROLLARY. $\mathrm{Cl}(M, w)\times I$, $C(\mathrm{Cl}(M, w))$ *and* $S(\mathrm{Cl}(M, w))$ *are all homeomorphic to* I^{n+1}.

PROOF OF THEOREM 1. Let $B=B(M, e)$, $e=ab$ and p be the barycenter of e. For each $x\in e$ let $D_x=[x \circ \mathrm{Lk}(M, e)]\cap B$. D_p divides B into two clusters C_a and C_b which are incident on D_p; C_a and C_b are piecewise linearly equivalent to $\mathrm{Cl}(M, a)$ and $\mathrm{Cl}(M, b)$, respectively.

Now let $U=C_a\cup C_a'-2D_p$ in $2B$. Clearly we also have $U\approx 2C_a-D_p\subseteq 2C_a\approx S^n$. By Lemma 4 for each $x\in ap-p$, $2D_x\approx S(\mathrm{Lk}(M, e))$; the latter is an $(n-1)$-pseudosphere by Lemma 8. It follows by Lemmas 4 and 7 that $U\approx E^n$. Since again by Lemma 4 $2D_p=\mathrm{Bd}\ U$ is bicollared in $2B$, U can be expanded to an open n-cell containing $C_a\cup C_a'$. Proposition II is now invoked to show us that $2B\approx S^n$.

LEMMA 9. *Let* $e=ab$ *be a* 1-*simplex in* K *and* p *be the midpoint of* e. *There is a homeomorphism of* $B({}^1\mathrm{Cl}(K, e), a)$ *onto the barrel neighborhood of* ${}^1(ap)$ *in* ${}^1[ap \circ \mathrm{Lk}(K, e)]$; *furthermore the map is the identity except possibly where it is defined in* $\mathrm{Cl}({}^1K, ap)$.

This map may be found by central projection through a in each maximal simplex of $\mathrm{Cl}(K, e)$.

PROOF OF THEOREM 2. This proceeds by induction on the number of vertices of T. It is obvious for one vertex by III.

Assume T_1 and T_2 are disjoint nonempty trees in T and e is an edge such that $T_1\cup e\cup T_2=T$. Let B, B_1 and B_2 be the barrel neighborhoods of ${}^1T$, ${}^1T_1$ and ${}^1T_2$, respectively, in ${}^1M$. By Lemma 9 the disjoint sets B_1 and B_2 can be stretched by homeomorphisms h_1 and h_2 so that $B=h_1(B_1)\cup h_2(B_2)$. Further $D=h_1(B_1)\cap h_2(B_2)\subseteq p \circ \mathrm{Lk}(M, e)$ where p is the midpoint of e. It may now be seen from examining the map described in Lemma 9 that D is cellular in both $2h_1(B_1)$ and $2h_2(B_2)$; or one can deduce this from Lemma 4. The rest of the proof resembles that of Theorem 1.

BIBLIOGRAPHY

1. J. W. Alexander, *The combinatorial theory of complexes*, Ann. of Math. (2) **31** (1930), 292–320.

2. M. Brown, *A proof of the generalized Schoenflies theorem*, Bull. Amer. Math. Soc. **66** (1960), 74–76.

3. B. Mazur, *On embeddings of spheres*, Bull. Amer. Math. Soc. **65** (1959), 59–65.

4. R. H. Rosen, *Stellar neighborhoods in polyhedral manifolds*, Proc. Amer. Math. Soc. (to appear).

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