ON A CONJECTURE CONCERNING PLANAR COVERINGS OF SURFACES

BY BERNARD MASKIT

Communicated by Lipman Bers, January 29, 1963

C. D. Papakyriakopoulos [1] recently proposed two conjectures in conjunction with his work on the Poincaré conjecture. We present here three counter examples to the second of these. He has subsequently modified his conjectures [2] as suggested by these examples. The conjecture to be contradicted is the following.

Let $S$ be a closed orientable surface of genus $g \geq 2$. Let $A_1, B_1, \ldots, A_g, B_g$ be a fundamental system of $S$ based at $o$ [1, p. 360]. Let $a_1, b_1, \ldots, a_g, b_g$ be the elements of $\pi_1(S, o)$ corresponding to $A_1, B_1, \ldots, A_g, B_g$ respectively; then

$$\pi_1(S, o) \cong F = \left( a_1, b_1, \ldots, a_g, b_g : \prod_{i=1}^{g} [a_i, b_i] \right).$$

Let $\phi$ be the free group freely generated by $a_1, b_1, \ldots, a_g, b_g$. Let $\tau_j$ be a word in the $a$'s and $b$'s representing an element of $[\phi, \phi], j = 1, \ldots, g$. Then for some subset $(m, \ldots, n)$ of $(1, \ldots, g)$ the regular covering surface $\tilde{S}$ of $S$, corresponding to $\langle [a_m, b_m\tau_m], \ldots, [a_n, b_n\tau_n] \rangle$ in $F$ [1, p. 361, footnote 5], is planar.

The three examples are differentiated by the following properties.

A. The elements $b_j\tau_j$ in $F$, $j = 1, \ldots, g$, can be represented by simple loops on $S$ [1, p. 365].

B. The words $b_j\tau_j$ in the $a$'s and $b$'s are cyclically reduced.

In all three examples we take $S$ of genus 2, with the basis $A_1, B_1, A_2, B_2$ as shown in Figure 1. For the first example we take $\tau_1 = [b_1^{-1}, b_2], \tau_2 = [b_2^{-1}, b_1]$; this satisfies A but not B. In the second example $\tau_1 = [b_2^{-1}, a_1^{-1}], \tau_2 = [b_1^{-1}, a_2^{-1}]$; this satisfies B but not A. In the third example $\tau_1 = [b_2, a_2], \tau_2 = [b_1, a_1]$; this satisfies both A and B.

We present here a proof only for the third counter example. The proofs for the first two are essentially the same except that, for these, one does not need the explicit construction of a certain group, and in the second example there are 19 cases to consider, while there are 7 cases in both the first and third.

We now assume that $\tilde{S}_1$, the regular covering surface of $S$ corresponding to $\langle [a_1, b_1\tau_1] \rangle \langle \tau_1 = [b_2, a_2] \rangle$, is planar. Let $C_1$ be a loop on

---

1 This research was supported by the Office of Naval Research under Contract No. NONR-285(46) and by the National Science Foundation. The author is currently a National Science Foundation Graduate Fellow.
$S$ representing $[a_1, b\tau_1]$ as shown in Figure 2, and let $C_2$ be a directed curve "parallel" to $C_1$. Let $o$ be the point of intersection of $C_1$ and $C_2$ marked in Figure 2. We now lift $C_1$ and $C_2$ to $\tilde{C}_1$ and $\tilde{C}_2$ respectively, starting at a point $\delta$ over $o$. Since $\tilde{S}_1$ is planar, $\tilde{C}_1$ and $\tilde{C}_2$ must have a second point of intersection $\tilde{P}$ which projects to a point of intersection $P$, of $C_1$ and $C_2$. If we orient $S$ and lift the orientation to $\tilde{S}_1$, then we can choose $\tilde{P}$ so that the sense of intersection at $\tilde{P}$ is the reverse of that at $\delta$, and by projection, the senses of intersection at $P$ and $o$ are reversed. Therefore $P$ must be one of the points marked 1, $\cdots$, 7 in Figure 2. Furthermore, if we follow $C_1$ from $o$ to $P$ and $C_2$ from $P$ to $o$, then the element of $F$ corresponding to this loop lies in the defining subgroup for $\tilde{S}_1$.

We now have seven cases to consider. If, for example, $P$ is the point marked 1, then the element of $F$ obtained by the above construction is $b_1^{-1}a_1b_1$. But $b_1^{-1}a_1b_1$ cannot be in $\langle [a_1, b\tau_1]\rangle$, since the element of $\phi$ corresponding to the word $b_1^{-1}a_1b_1$ does not belong to $[\phi, \phi]$. The same reasoning shows that $P$ cannot be any of the points marked 2, $\cdots$, 6. Hence $P$ must be the point marked 7. Therefore the above construction gives us that $\tau_1 \in \langle [a_1, b\tau_1]\rangle$, i.e. $\tau_1$ belongs to the smallest normal subgroup of $F$ containing $[a_1, b\tau_1]$.

Nothing in the above is changed if we replace $\tilde{S}_1$ by $\tilde{S}_2$, the regular covering surface corresponding to $\langle [a_2, b\tau_2]\rangle$. Also if we look at Figure 2 upside down, the above construction shows that if $\tilde{S}_8$, corresponding to $\langle [a_2, b\tau_2]\rangle$, is planar, then $\tau_2 = \tau_1^{-1}$ is in $\langle [a_2, b\tau_2]\rangle$.

The relation, $\tau_1 \in \langle [a_1, b\tau_1]\rangle$, implies that $\tau_1$ must be the identity in the group.
$G = (a_1, b_1, a_2, b_2; [a_1, b_1][a_2, b_2], [a_1, b_1^2], [a_2, b_2^2])$

where $\tau_1 = [b_2, a_2]$ and $\tau_2 = [b_1, a_1]$. Let us now consider a group $\Gamma$ of $2 \times 2$ matrices on generators

$$
\begin{align*}
\alpha_1 &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, & \beta_1 &= \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, & \alpha_2 &= \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}, & \beta_2 &= \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix}
\end{align*}
$$

and the map $G \rightarrow \Gamma$ defined by

$$
\begin{align*}
a_1 &\rightarrow \alpha_1, & b_1 &\rightarrow \beta_1, & a_2 &\rightarrow \alpha_2, & b_2 &\rightarrow \beta_2.
\end{align*}
$$

This is a homomorphism, since

$$
[[\alpha_1, \beta_1] [\alpha_2, \beta_2]] = 1, \quad \alpha_1 = \beta_1 [\beta_2, \alpha_2], \quad \alpha_2 = \beta_2 [\beta_1, \alpha_1]
$$

as one can easily see. However, $\tau_1 \rightarrow [\beta_2, \alpha_2] \neq 1$. We have arrived at a contradiction.

The assumption that $\mathcal{S}_1$ is planar leads to a contradiction. Hence $\mathcal{S}_1$ is not planar.

The author wishes to thank Professor L. Bers and Dr. Papakyriakopoulos for many informative conversations.

**REFERENCES**


NEW YORK UNIVERSITY