ON A CONJECTURE CONCERNING PLANAR COVERINGS OF SURFACES

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C. D. Papakyriakopoulos [1] recently proposed two conjectures in conjunction with his work on the Poincaré conjecture. We present here three counter examples to the second of these. He has subsequently modified his conjectures [2] as suggested by these examples. The conjecture to be contradicted is the following.

Let S be a closed orientable surface of genus $g \geqq 2$. Let $A_1, B_1, \cdots, A_g, B_g$ be a fundamental system of S based at o [1, p. 360]. *Let $a_1, b_1, \cdots, a_g, b_g$ be the elements of $\pi_1(S, o)$ corresponding to $A_1, B_1, \cdots, A_g, B_g$ respectively; then*

$$\pi_1(S, o) \simeq F = \left(a_1, b_1, \cdots, a_g, b_g: \prod_{i=1}^{g} [a_i, b_i].\right)$$

Let ϕ be the free group freely generated by $a_1, b_1, \cdots, a_g, b_g$. Let τ_j be a word in the a's and b's representing an element of $[\phi, \phi]$, $j = 1, \cdots, g$. Then for some subset $(m, \cdots, n)$ of $(1, \cdots, g)$ the regular covering surface $\tilde{S}$ of S, corresponding to $\langle [a_m, b_m\tau_m], \cdots, [a_n, b_n\tau_n]\rangle$ in F [1, p. 361, footnote 5], *is planar.*

The three examples are differentiated by the following properties.

A. The elements $b_j\tau_j$ in F, $j = 1, \cdots, g$, can be represented by simple loops on S [1, p. 365].

B. The words $b_j\tau_j$ in the a's and b's are cyclically reduced.

In all three examples we take S of genus 2, with the basis A_1, B_1, A_2, B_2 as shown in Figure 1. For the first example we take $\tau_1 = [b_1^{-1}, b_2]$, $\tau_2 = [b_2^{-1}, b_1]$; this satisfies A but not B. In the second example $\tau_1 = [b_2^{-1}, a_1^{-1}]$, $\tau_2 = [b_1^{-1}, a_2^{-1}]$; this satisfies B but not A. In the third example $\tau_1 = [b_2, a_2]$, $\tau_2 = [b_1, a_1]$; this satisfies both A and B.

We present here a proof only for the third counter example. The proofs for the first two are essentially the same except that, for these, one does not need the explicit construction of a certain group, and in the second example there are 19 cases to consider, while there are 7 cases in both the first and third.

We now assume that $\tilde{S}_1$, the regular covering surface of S corresponding to $\langle [a_1, b_1\tau_1]\rangle$ $(\tau_1 = [b_2, a_2])$, *is planar*. Let C_1 be a loop on

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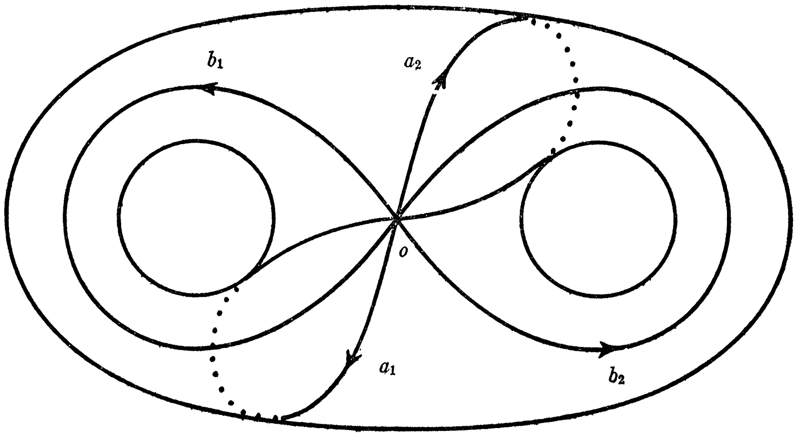

FIGURE 1

S representing $[a_1, b_1\tau_1]$ as shown in Figure 2, and let C_2 be a directed curve "parallel" to C_1. Let o be the point of intersection of C_1 and C_2 marked in Figure 2. We now lift C_1 and C_2 to $\tilde{C}_1$ and $\tilde{C}_2$ respectively, starting at a point $\tilde{o}$ over o. Since $\tilde{S}_1$ is planar, $\tilde{C}_1$ and $\tilde{C}_2$ must have a second point of intersection $\tilde{P}$ which projects to a point of intersection P, of C_1 and C_2. If we orient S and lift the orientation to $\tilde{S}_1$, then we can choose $\tilde{P}$ so that the sense of intersection at $\tilde{P}$ is the reverse of that at $\tilde{o}$, and by projection, the senses of intersection at P and o are reversed. Therefore *P must be one of the points marked* $1, \cdots, 7$ in Figure 2. Furthermore, *if we follow C_1 from o to P and C_2 from P to o, then the element of F corresponding to this loop lies in the defining subgroup for $\tilde{S}_1$.*

We now have seven cases to consider. If, for example, P is the point marked 1, then the element of F obtained by the above construction is $b_1^{-1}a_1b_1$. But $b_1^{-1}a_1b_1$ cannot be in $\langle[a_1, b_1\tau_1]\rangle$, since the element of ϕ corresponding to the word $b_1^{-1}a_1b_1$ does not belong to $[\phi, \phi]$. The same reasoning shows that P cannot be any of the points marked $2, \cdots, 6$. Hence P must be the point marked 7. Therefore the above construction gives us that $\tau_1 \in \langle[a_1, b_1\tau_1]\rangle$, i.e. *$\tau_1$ belongs to the smallest normal subgroup of F containing* $[a_1, b_1\tau_1]$.

Nothing in the above is changed if we replace $\tilde{S}_1$ by $\tilde{S}_2$, the regular covering surface corresponding to $\langle[a_1, b_1\tau_1], [a_2, b_2\tau_2]\rangle$. Also if we look at Figure 2 upside down, the above construction shows that if $\tilde{S}_3$, corresponding to $\langle[a_2, b_2\tau_2]\rangle$, is planar, then $\tau_2 = \tau_1^{-1}$ is in $\langle[a_2, b_2\tau_2]\rangle$.

The relation, $\tau_1 \in \langle[a_1, b_1\tau_1]\rangle$, implies that *$\tau_1$ must be the identity in the group*

$$G = (a_1, b_1, a_2, b_2: [a_1, b_1][a_2, b_2], [a_1, b_1\tau_1], [a_2, b_2\tau_2])$$

where $\tau_1 = [b_2, a_2]$ and $\tau_2 = [b_1, a_1]$. Let us now consider a group Γ of 2×2 matrices on generators

$$\alpha_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \beta_1 = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \quad \alpha_2 = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}, \quad \beta_2 = \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix}$$

and the map $G \to \Gamma$ defined by

$$a_1 \to \alpha_1, \qquad b_1 \to \beta_1, \qquad a_2 \to \alpha_2, \qquad b_2 \to \beta_2.$$

This is a homomorphism, since

$$[\alpha_1, \beta_1][\alpha_2, \beta_2] = 1, \qquad \alpha_1 = \beta_1[\beta_2, \alpha_2], \qquad \alpha_2 = \beta_2[\beta_1, \alpha_1]$$

as one can easily see. However, $\tau_1 \to [\beta_2, \alpha_2] \neq 1$. We have arrived at a contradiction.

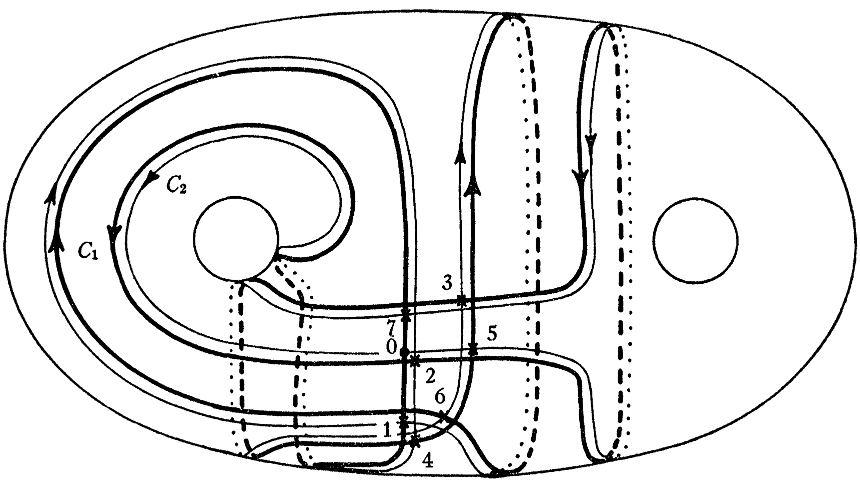

FIGURE 2

The assumption that $\tilde{S}_1$ is planar leads to a contradiction. Hence $\tilde{S}_1$ *is not planar.*

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2. ———, *A reduction of the Poincaré conjecture to other conjectures.* II, Bull. Amer. Math. Soc. **69** (1963), 399–401.

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