The object of Theorem 1 below is to establish the existence of periodic solutions of an autonomous differential equation \( \dot{y} = f(y) \) by an extension of the Poincaré method of sections (see [2; 4]). The following situation is envisaged: the equation is defined on a subset \( D \) of euclidean space and has unique solutions \( y(x, t) \) jointly continuous in \( t \) and the initial point \( x \); \( D \) contains a compact subset \( K \) with the property that the positive trajectories starting from points of \( K \) remain in \( K \). The assignment to \( x \) in \( K \) and \( t \) in \([0, \infty) \) of the point \( T_t(x) = y(x, t) \) in \( K \) defines a continuous one-parameter semigroup \( T_t \) acting on \( K \), i.e., \( T_t \) is jointly continuous in \( x \) and \( t \), \( T_0 \) is the identity on \( K \) and \( T_{t+s} = T_t \circ T_s \).

**Theorem 1.** Let \( K \) be a connected finite complex, let \( T_t \) be a continuous one-parameter semigroup acting on \( K \) and let \( \omega \) be a closed 1-form on \( K \) (defined over a portion of euclidean space containing \( K \)) with integer-valued periods. Make the following two assumptions on \( K \), \( T_t \) and \( \omega \):

A. For each \( x \) in \( K \) there is a \( t \) for which the integral of \( \omega \) over the trajectory from \( x \) to \( T_t(x) \) is positive.

B. The classes of closed paths in \( K \) over which the integral of \( \omega \) vanishes form a subgroup of the fundamental group of \( K \). Assume that the corresponding covering space \( K' \) has nonvanishing Euler characteristic.

**Conclusion:** \( T_t \) has a periodic trajectory, i.e., there is an \( x \) in \( K \) and a period \( p > 0 \) such that \( T_{t+p}(x) = T_t(x) \) for all \( t \geq 0 \).

**Remark a.** If we denote the integral of \( \omega \) over the trajectory from \( x \) to \( T_t(x) \) by \( \Delta(x, t) \), assumption A implies that there exists a positive constant \( a \) such that \( at < \Delta(x, t) \) for sufficiently large \( t \). Thus \( \Delta(x, t) \) converges uniformly to \( +\infty \). If \( T_t \) is engendered by the differential equation \( \dot{y} = f(y) \), \( \Delta(x, t) \) can be written as the integral with respect to \( t \) of the scalar product \( \omega \cdot f \), evaluated along the trajectory from \( x \) to \( T_t(x) \).

**Remark b.** Although the covering space \( K' \) is not a finite complex, assumption A implies that \( K' \) has finite Betti numbers, so that its Euler characteristic is defined.

**Remark c.** The period of the periodic trajectory disclosed by the theorem is bounded by a number depending on the uniform rate of
convergence of $\Delta(x, t)$ to $+\infty$, the periods of $\omega$ and the Betti numbers of $K'$.

By means of the construction outlined in [2], Theorem 1 can be derived from the following theorem.

**Theorem 2.** Let $F$ be an upper semicontinuous multiple-valued function from a finite complex $X$ into itself. Let the system of endomorphisms $F_{*p}$ of $H_p(X)$, the homology groups of $X$ with real coefficients, be induced by $F$. Denote by $r_p$ the lowest value to which rank $F_{*p}$ descends as $k$ increases. Then $\sum (-1)^n r_p \neq 0$ implies that $F$ has a periodic point: $x \in F^n(x)$ and the period $N$ does not exceed the larger of $\sum r_{2q}$ and $\sum f_{2q+1}$.

The proof of Theorem 2, using the Lefschetz formula for multiple-valued functions [4; 5] is essentially the same as that of the more special theorem in [1].

The relationships in Theorem 1 are illustrated by the following construction. Let $f$ be any continuous mapping of a connected finite complex $X$ into itself. The mapping cylinder $C_f$ of $f$, constructed using two copies of $X$, one for the domain and one for the range of $f$, can be made into a space $K$ by identifying the two copies. A semigroup $T_t$ acting on $K$ is obtained by moving all points at a uniform rate along the segments from $x$ to $f(x)$. A closed 1-form $\omega$ with integer periods can be defined on $K$ which is zero on the subspace $X$ and such that the integral of $\omega$ over any segment from $x$ to $f(x)$ is $+1$; $\omega$ satisfies assumption A. The covering space $K'$ is then a space obtained by coupling together copies $C_n$, $-\infty < n < +\infty$, of $C_f$. For the endomorphism $f_{*p}$ of $H_p(X)$, the integer $r_p$ defined in the statement of Theorem 2 turns out to be the $p$th Betti number of $K'$, so that by Theorem 2 nonvanishing of the Euler characteristic of $K'$ (assumption B) implies that $f$ has a periodic point and $T_t$ a periodic trajectory.

A proof of Theorem 1 will appear elsewhere.

**References**


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