ON THE RATE OF GROWTH OF ENTIRE FUNCTIONS OF FAST GROWTH

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Communicated by R. C. Buck, January 29, 1963

1. Introduction. The purpose of this note is to generalize the following well-known formula to give the order \( \rho \) and type \( \sigma \) of an entire function \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) with \( M(r) = \max_{|z| = r} |f(z)| \), that is [1; 2],

\[
\rho = \limsup_{r \to \infty} \frac{\log \log M(r)}{\log r} = \limsup_{n \to \infty} \frac{n \log n}{-\log |a_n|}, \tag{1}
\]

\[
\sigma = \limsup_{r \to \infty} \frac{\log M(r)}{r^\rho} = \frac{1}{\log \log M(r)} \limsup_{n \to \infty} n |a_n|^{r/\rho}. \tag{2}
\]

It will be observed that the coefficient \( 1/(ep) \) in (2) comes exclusively into the case of entire functions of finite order as we will see in the Theorem I.


**NOTATION 1.** \( \exp[0]^m = \log[0]^m = x, \exp[\pm 1]^m = \log[\pm 1]^m = \exp(\exp[\pm 1]^m x) = \log(\log[\pm 1]^m x) \) \( (m = 0, \pm 1, \pm 2, \cdots) \).

**NOTATION 2.**

\( E_{[r]}(x) = \prod_{i=0}^{r} \exp[i] x, \quad \Lambda_{[r]}(x) = \prod_{i=0}^{r} \log[i] x, \)

\( E_{[-1]}(x) = x/\Lambda_{[-1]}(x), \quad \Lambda_{[-1]}(x) = x/E_{[-1]}(x), \)

\( x = E_{[r]}^{-1}(y) \iff y = E_{[r]}(x) \quad (r = 0, \pm 1, \pm 2, \cdots). \)

**LEMMAS.** The functions \( \exp[m] x, \log[m] x, E_{[r]}(x), \Lambda_{[r]}(x), E_{[-1]}^{[r]}(x) \) \( (m = 0, \pm 1, \pm 2, \cdots; r = 0, 1, 2, \cdots) \) all increase monotonically and we have

\[
\frac{d}{dx} (\exp[m] x) = \frac{E_{[m]}(x)}{x} = \frac{1}{\Lambda_{[-m-1]}(x)}, \tag{3}
\]

\[
\frac{d}{dx} (\log[m] x) = \frac{1}{\Lambda_{[m-1]}(x)} = \frac{E_{[-m]}(x)}{x} \quad (m = 0, \pm 1, \pm 2, \cdots) \tag{4}
\]

\[
E_{[r]}^{[-1]}(y) = \begin{cases} y & (r = 0) \\ \log^{[r-1]}(\log y - \log^{[2]} y + O(\log^{[3]} y))(r = 1, 2, 3, \cdots) \end{cases} \tag{5}
\]
5. Formulas for \( \lambda(q) \) and \( \kappa(q) \).

**Theorem 1.** Let \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) be a transcendental entire function of index \( q \), then \( \lambda(q) = \lambda \) and \( \kappa(q) = \kappa \) of \( f(z) \) is given by \( \lambda = \mu \) and \( \kappa = \tau \) where

\[
\mu = \limsup_{n \to \infty} \frac{n \log^{[q-1]} \mu}{-\log |a_n|} \quad (q = 2, 3, 4, \ldots)
\]

and

\[
\tau = \begin{cases} 
(1/e\lambda) \cdot \limsup_{n \to \infty} n |a_n|^{\lambda/n} & (q = 2) \\
\limsup_{n \to \infty} \log^{[q-1]} \mu \cdot |a_n|^{\lambda/n} & (q = 3, 4, 5, \ldots).
\end{cases}
\]

**Proof.** From \( -n \log^{[q-1]} \mu/\log |a_n| \leq \mu + \epsilon \), we have, with \( S = \exp^{[q-1]}((2r)^{\mu+\epsilon}) \), that

\[
M(r) \leq \sum_{n \leq s} |a_n| r^n + \sum_{n > s} |a_n| r^n
\]

\[
\leq \exp^{[q-1]}((2r)^{\mu+\epsilon}) \cdot \sum_{n=0}^{\infty} (\log^{[q-1]} n)^{n/((\mu+\epsilon))} + \sum_{n=0}^{\infty} \frac{1}{2^n}
\]

\[
= O(\exp^{[q-1]} r^{\mu+\epsilon}).
\]
Letting $\epsilon \to 0$ we have $\lambda \leq \mu$.
Let $\sigma = e\lambda r$ for $q = 2$ and $\sigma = r$ for $q \geq 3$, from
\begin{equation}
|a_n|^{\lambda/n} \cdot \log^{[q-2]} n \leq \sigma + \epsilon
\end{equation}
we see, by logarithmic differentiation with (4), that the maximum of $|a_n|/r^n$ is estimated by
\begin{equation}
|a_n|/r^n \leq \exp\left(\exp^{[q-2]}\left(\frac{(\sigma + \epsilon)\lambda}{\exp(E_{[1-q]}(n))}\right) \cdot \frac{E_{[1-q]}(n)}{\lambda}\right) = \phi(r).
\end{equation}
Hence, we have, with $s' = \exp^{[q-2]}((\sigma + 2\epsilon)r^\lambda)$, using (6), that
\begin{equation}
M(r) \leq \sum_{n \leq r'} |a_n|/r^n + \sum_{n > r'} |a_n|/r^n
\end{equation}
\begin{align*}
&\leq s'\phi(r) + \sum_{n=0}^{\infty} \left(\frac{\sigma + \epsilon}{\sigma + 2\epsilon}\right)^n
\end{align*}
\begin{align*}
&= O(\exp^{[q-1]}((r + 3\epsilon)r^\lambda)).
\end{align*}
Letting $\epsilon \to 0$, we have $\kappa \leq \tau$. Suppose now, that $M(r) < C \exp^{[q-1]}((\kappa + \epsilon)r^\lambda)$ then $|a_n| < M(r)/r^n$ is estimated by minimizing its right hand side which occurs, by (3), at $r = (E_{[q-2]}(n/\lambda)/(\kappa + \epsilon))^{1/\lambda}$. Hence, we have
\begin{equation}
|a_n| < \frac{C(\kappa + \epsilon)^{\lambda/n} \cdot \exp^{[q-1]}(E_{[q-2]}^{[-1]}(n/\lambda))}{(E_{[q-2]}^{[-1]}(n/\lambda))^{\lambda/n}}
\end{equation}
from which we have by (5) and (7), that
\begin{equation}
\lambda \geq \limsup_{n \to \infty} n \log^{[q-1]} n \frac{n \log^{[q-1]} n}{-\log |a_n|} = \mu
\end{equation}
and
\begin{equation}
\kappa + \epsilon \geq \frac{\tau}{\sigma} \cdot \limsup_{n \to \infty} \log^{[q-2]} n \cdot |a_n|^{\lambda/n} = \tau.
\end{equation}
The theorem is thereby proved.

4. Further remarks.
1. Utterly integer valued transcendental entire function. We have many results on the integer valued entire functions of index $q = 2$, (finite order) i.e., [3] but here we introduce a theorem on index $q \geq 3$, whose proof together with its generalization and applications on number theory will appear in a future paper.

Theorem II. A transcendental entire function which together with all its derivatives assumes integers at all integer points (utterly integer
valued) must have index \( q \geq 3 \) and, if the index is 3, then its rank must be \( \lambda(3) \geq 1 \). This estimation is the best possible one, since there exist such a transcendental entire function of index 3 and \( \lambda(3) = 1 \).

2. Entire function of infinite index. For any positive increasing function \( \psi(n) \) with \( \psi(n) \to \infty \) as \( n \to \infty \), but for no function with \( \lim \inf_{n \to \infty} \psi(n) = m < \infty \), the series \( f(z) = \sum_{n=0}^{\infty} z^n/(\psi(n))^n \) represents an entire function, hence if \( \psi(n) \) grows slower than any \( \log^{[N]} n \) with fixed \( N \), then \( f(z) \) represents an entire function of infinite index. To define the rate of growth, the natural comparison function will be \( \phi(r) = f((\alpha + \varepsilon)r^\theta) \) with \( f(x) = \exp[\varepsilon] \), \( \lceil x \rfloor \): Gauss step function.

3. Entire functions of nonintegral index. Consider \( f(x) = \exp^{(q/p)} x \) as a well defined solution of a simultaneous functional equation \( \exp^{(p)}(f(x)) = \exp^{(q)} x \) (\( l = 0, \pm 1, \pm 2, \cdots \)) and for real \( r \), define \( \exp^{(r)}(x) \) by uniform limit process. Generalize an index as the least number \( \eta \) such that for any given \( \varepsilon > 0 \), there exist \( r_0(\varepsilon) \) by which it satisfies \( M(r) < \exp^{(r+\varepsilon)}(r) \) for \( r \geq r_0(\varepsilon) \), when \( \eta < \infty \), define \( \lambda(q) \) and \( \kappa(q) \) by the similar manner.

The author conjectures to have the similar formula as in Theorem I, but this formulation is incomplete at this moment.

5. A research problem. To generalize the discussion into the meromorphic functions, we propose the following problem which is originally given by E. G. Straus.

**Problem.** Let \( f(z) \) be a meromorphic function and \( T(r) \) be its characteristic function, let

\[
\lambda(q) = \limsup_{r \to \infty} \frac{\log^{[q-1]} T(r)}{\log r} = \lambda
\]

and, when \( 0 < \lambda < \infty \),

\[
\kappa(q) = \limsup_{r \to \infty} \frac{\log^{[q-1]} T(r)}{r^\lambda} = \kappa.
\]

Find the formula to give \( \lambda \) and \( \kappa \) from the Taylor series coefficients of \( f(z) \).

**REFERENCES**


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