RESEARCH PROBLEMS

3. Richard Bellman: *Degrees of noncommutativity.*

1. Let X_1 and X_2 be two complex $N \times N$ matrices which do not commute. Introduce the norm of an $N \times N$ matrix Z by $\|Z\|^2 = \operatorname{tr}(Z\bar{Z}')$. What is the minimum of $\|X_1 - Y_1\|^2 + \|X_2 - Y_2\|^2$ over all $N \times N$ matrices Y_1 and Y_2 which do commute? Is there a general inequality connecting $\|X_1 - Y_1\|^2$, $\|X_2 - Y_2\|^2$, $\|X_1X_2 - X_2X_1\|$, $\|Y_1Y_2 - Y_2Y_1\|$? Extend the problem by means of different norms and by considerations of various types of operators in Hilbert space.

2. Let X_1 and X_2, as before, not commute. What is the minimum value over all complex scalars a_i of $\|X_2 - \sum_{k=0}^{N-1} a_k X_1^k\|^2$, and how does the minimum behave as a function of N as $N \to \infty$?

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4. Albert A. Mullin: *Some related number-theoretic functions.*

The *Fundamental theorem of arithmetic* provides a unique decomposition of every natural number greater than 1 into a product of powers of distinct primes, the prime bases being arranged in their natural linear ordering. Consider such a decomposition. E.g., $48 = 2^4 \cdot 3$. Suppose that those exponents greater than 1, if any, are also expanded in the same way and that the exponents greater than 1, if any, of those exponents are so expanded, etc., until the process terminates as it always does in a finite number of steps. Call the final configuration consisting of primes alone a *mosaic.* Clearly the mosaic is different, in general, from the usual multiplicatively linear array of primes alone. E.g., the mosaic of 48 is $2^{2^2} \cdot 3$. There is a one-one effectively calculable map ν from the natural numbers onto the mosaics (identify the "empty" mosaic with 1), which is an alternate formulation of FTA.

For any mosaic take the product of the primes alone that appear in it, thereby yielding a unique natural number called the *residuum* of the mosaic. E.g., the residuum of the mosaic of 48 is $2 \cdot 2 \cdot 2 \cdot 3 = 24$. Let ρ be the map from mosaics to the natural numbers associated with residua. Put $\psi = \rho(\nu(\cdot))$. Clearly ψ is an "interesting" integer-valued number-theoretic and algebraic map from the natural numbers *onto* the natural numbers. E.g., $\psi(a \cdot b) \gtreqless \psi(a) \cdot \psi(b)$ with equality if $\gcd\{a, b\} = 1$; i.e., ψ is multiplicative. Also $\psi(n) \leqq n$, for every natural number n. Further square-free natural numbers, among others, are invariant under ψ. Also $\psi(k) = k$ for infinitely many k and $\psi(m) \neq m$ for infinitely many m. In addition ψ is an effectively calculable function. Since ψ is a finitely many-one map from the natural numbers *onto* the natural numbers one can define another effectively calculable number-theoretic function ξ by $\xi(n) = \text{cardinality } \{\psi^{-1}(n)\}$, where n is a natural number.

What are the number-theoretic and algebraic invariance characteristics of ξ? Is ξ an *onto* mapping? If one defines $\psi^2 = \psi(\psi(\cdot))$ and ψ^k, $k \geqq 3$, recursively, does there exist an *explicit* algorithm that determines, for each natural number n, the least natural number t such that $\psi^t(n) = \psi^{t+m}(n)$ for every natural number m? If not is there a "simple" bound on it, e.g., $t \leqq n$?

Now define $\alpha(n) = \text{card}\{a \in N : a \leqq n,\ n \in N,\ \psi(a) = a\}$ and $\beta(n) = \text{card}\{a \in N : a \leqq n,\ n \in N,\ \psi(a) \neq a\}$ where N is the set of all natural numbers. Let $D(n)$ be the distribution function for the square-free integers. Clearly $D(n) \leqq [\alpha(n)/n] < [\alpha(n)/\beta(n)]$ for every $n \in N$. Further it can be shown that $6/\pi^2 = \lim_{n\to\infty} D(n) < \liminf_{n\to\infty} [\alpha(n)/n] \leqq \limsup_{n\to\infty} [\alpha(n)/n] < 1$ and $\lim_{n\to\infty} D(n) < \liminf_{n\to\infty} [\alpha(n)/\beta(n)] \leqq \limsup_{n\to\infty} [\alpha(n)/\beta(n)] < \infty$. Also $\lim_{n\to\infty} [\alpha(n)/n]$ and $\lim_{n\to\infty} [\alpha(n)/\beta(n)]$ exist. Do any *double*-limit theorems exist?

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5. R. S. L. Srivastava: *Function theory.*

What necessary and sufficient conditions should be satisfied by the coefficients a_n in a power series $\sum_0^\infty a_n z^n = f(z)$ so that $f(z)$ will be an entire function, real for real z, and having no real zeros?

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