INFINITELY REPEATED MATRIX GAMES FOR WHICH PURE STRATEGIES SUFFICE

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1. **Introduction.** Let $A = \|a(i, j)\|$ be an $r \times s$ matrix with real entries. Consider the game in which nature picks a column, j, the experimeter a row, i, and the experimenter is paid a sum $a(i, j)$ (possibly negative). The game is to be repeated countably many times, with the restriction that nature must select a sequence with averages. That is, for each j, $j = 1, \cdots, s$, the frequency with which the column j is chosen in the first n plays, $q_j(n)$, converges, as $n \to \infty$, to some q_j.

Hannan [2] has exhibited a mixed strategy for the experimenter such that, for every sequence of nature with frequencies q_j, the average expected payoff will converge to $M = \max_i \sum_{j=1}^{s} a(i, j) q_j$. Blackwell [1] has exhibited a strategy such that, for every sequence of nature with frequencies q_j, $\lim_{N\to\infty} (1/N) \sum_{n=1}^{N} P_n = M$ with probability one, where P_n denotes the payoff at time n under the chosen mixed strategy.

We here exhibit a class of *pure* strategies under which the averages $(1/N) \sum_{n=1}^{N} P_n$ converge to M for every allowable sequence of nature. (By a pure strategy we mean a function $f(\{x_n\}) = \{y_n\}$ where $\{x_n\}$ is a sequence of elements of $\{1, \cdots, s\}$ and $\{y_n\}$ is a sequence of elements of $\{1, \cdots, r\}$ with y_n constant on $\{x_1, \cdots, x_{n-1}\}$ cylinders. In brief, the experimenter's choice at time n is a function of nature's choices at times $1, 2, \cdots, n-1$.) Our result insures that, without the necessity of mixed strategies by the experimenter, but with a suitably chosen pure strategy, his average payoff will converge to the minimax payoff if nature chooses a minimax mixed strategy and, moreover, will take full advantage of any weaker strategy on nature's part.

2. **Example.** Let nature select a sequence of zeros and ones with a density, d, of ones. The experimenter, after trial n, having observed the past, guesses nature's choice at time $n+1$ and is awarded 1 or 0 units according as he is right or wrong; i.e., the payoff matrix is

$$\begin{Vmatrix} 1 & 0 \\ 0 & 1 \end{Vmatrix}.$$

A strategy "succeeds" when its average payoff approaches $\max(d, 1-d)$. The strategy of always guessing 1 fails when $d < \frac{1}{2}$;

the strategy of guessing, at time $n+1$, the majority up to time n (with ties decided somehow) fails against some sequences with $d=\frac{1}{2}$. One successful strategy is to guess, for all n such that $2^i<n\leqq 2^{i+1}$, the majority up to time 2^i. The theorem below generalizes this scheme to arbitrary finite payoff matrices.

3. Main result.

THEOREM. *Let $A=\|a(i,j)\|$ be an $r\times s$ matrix of real numbers. Let $S=\{1,\cdots,s\}$ and let $\{x_i \mid i=1,2,\cdots\}$ be a sequence of elements of S such that if $Q_j(m,n)=\text{crd}\,\{x_i \mid x_i=j,\ m<i\leqq n\}$ then*

$$\lim_{n\to\infty}\frac{Q_j(0,n)}{n}=q_j. \tag{1}$$

Let $\{n_k \mid k=1,2,\cdots\}$ be an increasing sequence of positive integers such that $n_1=1$ and such that $\lim\inf_k n_{k+1}/n_k>1$. Given k, let $i(n_k)$ be the least integer i which maximizes $\sum_{j=1}^{s}a(i,j)Q_j(0,n_k)$. Define $y_1=1$, and, if $n_k<n\leqq n_{k+1}$, let $y_n=i(n_k)$. Then

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}a(y_n,x_n)=M=\max_i\sum_{j=1}^{S}a(i,j)q_j. \tag{2}$$

LEMMA 1. *Let $\{a_k\}$, $\{b_k\}$, $k=1,2,\cdots$, be given, with $b_k>0$ for all k. Let $A_n=\sum_{k=1}^{n}a_k$, $B_n=\sum_{k=1}^{n}b_k$. Then:*

(a) If $\lim_{n\to\infty}B_n=\infty$, and if $\lim_{k\to\infty}a_k/b_k=K<\infty$, then $\lim_{n\to\infty}A_n/B_n=K$.

(b) If $\lim\sup_k B_k/b_k<\infty$, and if $\lim_{n\to\infty}A_n/B_n=K<\infty$, then $\lim_{k\to\infty}a_k/b_k=K$.

LEMMA 2. *Let $\{b_k\}$, $k=1,2,\cdots$ be given, with $b_k>0$ for all k, such that $B_n\to\infty$, and let $f(n)$ be a real-valued function of n. Given $n>B_2$, select $k=k(n)$ such that $B_k<B_{k+1}<n\leqq B_{k+2}$. Then:*

(a) If $\lim_{n\to\infty}(f(n)-f(B_k))(n-B_k)^{-1}=\lim_{m\to\infty}f(B_m)/B_m=K<\infty$, then $\lim_{n\to\infty}f(n)/n=K$.

(b) If $\lim\sup_{k\to\infty}B_k/b_k<\infty$, and if $\lim_{n\to\infty}f(n)/n=K<\infty$, then $\lim_{n\to\infty}(f(n)-f(B_k))(n-B_k)^{-1}=K$.

We omit the proofs of the lemmas.

PROOF OF THE THEOREM. The proof is divided into two parts.

PART 1. We show $\lim_{k\to\infty}(1/n_k)\sum_{n=1}^{n_k}a(y_n,x_n)=M$. Since $\sum_{n=1}^{n_k}a(y_n,x_n)=a(1,x_1)+\sum_{l=1}^{k-1}\sum_{j=1}^{s}a(i(n_l),j)Q_j(n_l,n_{l+1})$, it suffices, by Lemma 1(a) to show that

$$\lim_{k\to\infty}\frac{1}{n_k-n_{k-1}}\sum_{j=1}^{s}a(i(n_{k-1}),j)Q_j(n_{k-1},n_k)=M. \tag{3}$$

But, by (1) and Lemma 1(b), for each j,

$$Q_j(n_{k-1}, n_k)(n_k - n_{k-1})^{-1} = Q_j(0, n_{k-1})(n_{k-1})^{-1} + \epsilon_j(k),$$

where $\lim_{k\to\infty} \epsilon_j(k) = 0$. Therefore, it suffices to prove:

$$\lim_{k\to\infty} \frac{1}{n_{k-1}} \sum_{j=1}^{s} a(i(n_{k-1}), j) Q_j(0, n_{k-1}) = M. \tag{4}$$

This is immediate from (1) and from the continuity of the function

$$F(z_1, \cdots, z_s) = \max_i \sum_{j=1}^{s} a(i, j) z_j.$$

PART 2. We show (2). If $n_k < n_{k+1} < n \leqq n_{k+2}$,

$$\sum_{i=1}^{n} a(y_i, x_i) = a(1, x_1) + \sum_{l=1}^{k} \sum_{j=1}^{s} a(i(n_l), j) Q_j(n_l, n_{l+1}) + \sum_{j=1}^{s} a(i(n_{k+1}), j) Q_j(n_{k+1}, n);$$

hence, by Lemma 2(a), it suffices to show

$$\lim_{n\to\infty} \frac{1}{n - n_k} \sum_{j=1}^{s} \{ a(i(n_k), j) Q_j(n_k, n_{k+1}) + a(i(n_{k+1}), j) Q_j(n_{k+1}, n) \} = M. \tag{5}$$

But

$$\frac{Q_j(n_{k+1}, n)}{n - n_k} = \frac{Q_j(n_k, n)}{n - n_k} - \frac{Q_j(n_k, n_{k+1})}{n - n_k} = \frac{Q_j(0, n_{k+1})}{n_{k+1}} + \delta_j(k) - \frac{n_{k+1} - n_k}{n - n_k} \left\{ \frac{Q_j(0, n_{k+1})}{n_{k+1}} + \eta_j(k) \right\},$$

where $\lim_{k\to\infty} \delta_j(k) = 0$ by Lemma 2(b) and $\lim_{k\to\infty} \eta_j(k) = 0$ by Lemma 1(b). Since, also,

$$\frac{Q_j(n_k, n_{k+1})}{n - n_k} = \frac{n_{k+1} - n_k}{n - n_k} \left\{ \frac{Q_j(0, n_k)}{n_k} + \zeta_j(k) \right\},$$

where $\lim_{k\to\infty} \zeta_j(k) = 0$, we have reduced the problem to showing that:

$$\lim_{n\to\infty} \left[\frac{n_{k+1} - n_k}{n - n_k} \sum_{j=1}^{s} \left\{ a(i(n_k), j) \frac{Q_j(0, n_k)}{n_k} - a(i(n_{k+1}), j) \frac{Q_j(0, n_{k+1})}{n_{k+1}} \right\} + \sum_{=1}^{s} a(i(n_{k+1}), j) \frac{Q_j(0, n_{k+1})}{n_{k+1}} \right] = M. \tag{6}$$

This follows from the continuity of F, as before. The proof of the theorem is complete.

References

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2. J. Hannan, *Approximation to Bayes risk in repeated play*, Annals of Mathematics Studies No. 39, Princeton Univ. Press, Princeton, N. J., 1957, pp. 97–139.

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TRANSVERSALITY IN MANIFOLDS OF MAPPINGS[1]

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1. **Introduction.** Let X and Y be differentiable manifolds and $\mathfrak{A}$ a space of mappings from X to Y. A common problem in differential topology is to approximate a mapping in $\mathfrak{A}$ by another in $\mathfrak{A}$ which is transversal to a given submanifold $W \subset Y$. Thus if $\mathfrak{A}_{X,W}$ is the subspace of mappings transversal to W it is important to know if $\mathfrak{A}_{X,W}$ is dense in $\mathfrak{A}$. Some famous examples are the Whitney immersion and embedding theorems [**8**] and the Thom transversality theorem [**4**; **7**]. In the next section we give sufficient conditions for density in case $\mathfrak{A}$ is a Banach manifold. The proof of the density theorem is indicated in the third section, and in the final section the Thom transversality theorem is obtained as a corollary.

2. **Density theorems.** Throughout this section X will be a manifold with boundary, Y and Z manifolds, $W \subset Y$ a submanifold (W, Y, Z without boundary) all of class C^r, $r \geqq 1$, and modelled on Banach spaces (see [**3**] for definitions).

2.1. DEFINITION. A C^r mapping $f: X \rightarrow Y$ is *transversal to W at a point* $x \in X$ iff either $f(x) \notin W$, or $f(x) = w \in W$ and there exists a neighborhood U of $x \in X$ and a local chart (V, ψ) at $w \in Y$ such that

$$\psi: V \rightarrow \boldsymbol{E} \times \boldsymbol{F}: V \cap W \rightarrow \boldsymbol{E} \times 0,$$

$\pi_1 \circ \psi$ is a diffeomorphism of $V \cap W$ onto an open set of $\boldsymbol{E}$, and $\pi_2 \circ \psi \circ f | U$ is a submersion [**3**, p. 20], where $\pi_1: \boldsymbol{E} \times \boldsymbol{F} \rightarrow \boldsymbol{E}$ and

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