This follows from the continuity of F, as before. The proof of the theorem is complete.

REFERENCES

1. D. Blackwell, *Controlled random walks*, Proceedings of the International Congress of Mathematicians, Vol. III, pp. 336–338, North-Holland, Amsterdam, 1956.
2. J. Hannan, *Approximation to Bayes risk in repeated play*, Annals of Mathematics Studies No. 39, Princeton Univ. Press, Princeton, N. J., 1957, pp. 97–139.

INSTITUTE FOR DEFENSE ANALYSES AND
UNIVERSITY OF CHICAGO

TRANSVERSALITY IN MANIFOLDS OF MAPPINGS[1]

BY RALPH ABRAHAM

Communicated by E. Spanier, January 14, 1963

1. **Introduction.** Let X and Y be differentiable manifolds and $\mathcal{A}$ a space of mappings from X to Y. A common problem in differential topology is to approximate a mapping in $\mathcal{A}$ by another in $\mathcal{A}$ which is transversal to a given submanifold $W \subset Y$. Thus if $\mathcal{A}_{X,W}$ is the subspace of mappings transversal to W it is important to know if $\mathcal{A}_{X,W}$ is dense in $\mathcal{A}$. Some famous examples are the Whitney immersion and embedding theorems [8] and the Thom transversality theorem [4; 7]. In the next section we give sufficient conditions for density in case $\mathcal{A}$ is a Banach manifold. The proof of the density theorem is indicated in the third section, and in the final section the Thom transversality theorem is obtained as a corollary.

2. **Density theorems.** Throughout this section X will be a manifold with boundary, Y and Z manifolds, $W \subset Y$ a submanifold (W, Y, Z without boundary) all of class C^r, $r \geqq 1$, and modelled on Banach spaces (see [3] for definitions).

2.1. DEFINITION. A C^r mapping $f: X \to Y$ is *transversal to W at a point* $x \in X$ iff either $f(x) \notin W$, or $f(x) = w \in W$ and there exists a neighborhood U of $x \in X$ and a local chart (V, ψ) at $w \in Y$ such that

$$\psi: V \to \boldsymbol{E} \times \boldsymbol{F}: V \cap W \to \boldsymbol{E} \times 0,$$

$\pi_1 \circ \psi$ is a diffeomorphism of $V \cap W$ onto an open set of $\boldsymbol{E}$, and $\pi_2 \circ \psi \circ f | U$ is a submersion [3, p. 20], where $\pi_1: \boldsymbol{E} \times \boldsymbol{F} \to \boldsymbol{E}$ and

[1] This work has been partially supported by the Office of Naval Research under contract Nonr(G)-00098-62 and the National Science Foundation under grant G19136.

$\pi_2: \boldsymbol{E}\times\boldsymbol{F}\rightarrow\boldsymbol{F}$ are the projections. The mapping f is *transversal to W on a subset* $K\subset X$, $f|K\pitchfork W$, iff f is transversal to W at every point $x\in K$; and f is *transversal to W*, $f\pitchfork W$, iff $f|X\pitchfork W$.

For some basic properties of transversality see Lang [**3**, p. 22]. Suppose $f: X\rightarrow Y$ and $g: Z\rightarrow Y$ are C^r mappings, and let $\Delta\subset Y\times Y$ denote the diagonal.

2.2. DEFINITION. The mappings $f: X\rightarrow Y$ and $g: Z\rightarrow Y$ are *transversal at points $x\in X$ and $z\in Z$* iff the product $f\times g: X\times Z\rightarrow Y\times Y$ is transversal to Δ at (x, z) in the sense of Definition 2.1. The mappings are *transversal on sets $K\subset X$ and $M\subset Z$*, $f|K\pitchfork g|M$, iff

$$f\times g\,|\,K\times M\pitchfork\Delta,$$

and they are *transversal*, $f\pitchfork g$, iff $f\times g\pitchfork\Delta$.

2.3. DEFINITION. A *C^r manifold of mappings* from X to Y is a set $\mathfrak{A}$ of C^r mappings from X to Y which is a C^r manifold such that the evaluation mapping

$$\mathrm{ev}: \mathfrak{A}\times X\rightarrow Y: (f, x)\rightarrow f(x)$$

is of class C^r.

For example it is known that if X is compact then the set $\mathcal{C}^r(X, Y)$ of all C^r mappings from X to Y has a natural structure of C^r manifold of mappings [**1**, p. 31; **2**; **5**].

If $K\subset X$ is any subset, and $\mathfrak{A}$ a manifold of mappings from X to Y, let $\mathfrak{A}_{K,W}\subset\mathfrak{A}$ be the subspace of mappings which are transversal to W on the set K. The following is easy to prove.

2.4. OPENNESS THEOREM. *If $K\subset X$ is a compact set, $W\subset Y$ a closed submanifold, and $\mathfrak{A}\subset\mathcal{C}^r(X, Y)$ a C^r manifold of mappings, then the subset $\mathfrak{A}_{K,W}$ is open in $\mathfrak{A}$.*

Recall that a *residual set* in a topological space is a countable intersection of open dense sets, a *Baire space* is one in which every residual set is dense, and by the Baire category theorem every Banach manifold is a Baire space.

2.5. DENSITY THEOREM. *Let X be an n-manifold with boundary, $K\subset X$ any subset, Y a Banach manifold (without boundary), and $W\subset Y$ a closed submanifold (without boundary) of finite codimension q, all of class C^r. Let $\mathfrak{A}\subset\mathcal{C}^r(X, Y)$ be a C^r manifold of mappings. If the evaluation map of $\mathfrak{A}$*

$$\mathrm{ev}: \mathfrak{A}\times X\rightarrow Y: (f, x)\rightarrow f(x)$$

is transversal to W on K and $r>\max(n-q, 0)$, then $\mathfrak{A}_{K,W}\subset\mathfrak{A}$ is residual.

This theorem is the main result, and may be generalized in several ways. For example suppose X, Y and Z are finite dimensional, $\mathcal{A}\subset\mathcal{C}^r(X, Y)$ and $\mathcal{B}\subset\mathcal{C}^r(Z, Y)$ are C^r manifolds of mappings, and $\text{ev}_{\mathcal{A}}$ and $\text{ev}_{\mathcal{B}}$ are the respective evaluation mappings. If $K\subset X$ and $M\subset Z$, let $\mathcal{A}\times\mathcal{B}_{K\times M}=\{(f, g)\in\mathcal{A}\times\mathcal{B}: f|K \pitchfork g|M\}$. Then the following are immediate.

2.6. COROLLARY. *If* $\text{ev}_{\mathcal{A}}|\mathcal{A}\times K \pitchfork \text{ev}_{\mathcal{B}}|\mathcal{B}\times M$ *and* $r>\max(\dim X+\dim Z-\dim Y, 0)$, *then* $\mathcal{A}\times\mathcal{B}_{K\times M}\subset\mathcal{A}\times\mathcal{B}$ *is residual.*

Let $\mathcal{B}=\{g\}$, a single map, and

$$\mathcal{A}_{K,g} = \{f\in \mathcal{A}: f|K\pitchfork g|Z\}.$$

2.7. COROLLARY. *If* $\text{ev}_{\mathcal{A}}|\mathcal{A}\times K\pitchfork g$ *and*

$$r > \max(\dim X + \dim Z - \dim Y, 0),$$

then $\mathcal{A}_{K,g}\subset\mathcal{A}$ *is residual.*

If in 2.7 g is an embedding and W its image, then 2.5 is obtained with the condition "W closed" deleted.

Note the symmetry of 2.2 and 2.6. Both may be extended to n-tuples of mappings having a common target, and the symmetry can be completed by allowing all sources to be manifolds with boundary, making only trivial modifications.

3. **Proof of the density theorem.** The Density Theorem 2.5 is proved from the following lemma by an easy point set argument.

DENSITY LEMMA. *If X has finite dimension n, $W\subset Y$ is a closed submanifold having finite codimension q, $\mathcal{A}\subset\mathcal{C}^r(X, Y)$ is a C^r manifold of mappings with $r>\max(n-q, 0)$, and the evaluation mapping is transversal to W at a point $(f, x)\in\mathcal{A}\times X$, then there exists a neighborhood $\mathcal{U}$ of $f\in\mathcal{A}$ and a neighborhood V of $x\in X$ such that $\mathcal{U}_{V,W}\subset\mathcal{U}$ is dense.*

If $f(x)\notin W$ the lemma is trivial. If $f(x)=w\in W$ the proof is immediate from these three propositions.

PROPOSITION A. *If the evaluation map is transversal to W at (f, x) and $f(x)=w\in W$, there are neighborhoods $\mathcal{U}$ of $f\in\mathcal{A}$ and V of $x\in X$ such that every point $g\in\mathcal{U}$ is contained in a p-dimensional submanifold Σ_g^p, $0\leq p\leq q$, such that $\text{ev}|\Sigma_g^p\times V\pitchfork W$.*

The proof of this proposition is prosaic, relying on techniques which have become standard since the publication of Lang's book [3].

For the second proposition suppose Σ^p is a p-dimensional submanifold of $\mathfrak{A}$, V an open set of X, and $\xi = \mathrm{ev} \mid \sum^p \times V$ is transversal to W. Then $W' = \xi^{-1}(W)$ is a submanifold of codimension q of $\sum^p \times V$. Let $\sigma: W' \to \sum^p$ denote the restriction to W' of the projection $\sum^p \times V \to \sum^p$.

Proposition B. *If σ is transversal to a point $f \in \sum^p$, then f is transversal to W on V.*

The proof of this proposition is a straightforward interpretation of the definitions.

The third proposition is a well-known theorem of Sard [6]. Recall that if $f: X \to Y$ is any C^1 mapping, a point $y \in Y$ is a *critical value* of f iff it is false that $f \pitchfork \{y\}$. Let Ω_f be the set of all critical values of f.

Proposition C (Sard). *If $f: \boldsymbol{R}^s \to \boldsymbol{R}^t$ is of class C^r with $r > \max(s-t, 0)$, then $\Omega_f \subset \boldsymbol{R}^t$ has outer measure zero.*

4. **The Thom transversality theorem.** Let X be a C^r manifold with boundary, Y a C^r manifold (without boundary), $\pi^k: J^k(X, Y) \to X \times Y$ the k-jet bundle of C^k maps from X to Y, $0 \leq k \leq r$, and

$$j^k: \mathfrak{C}^r(X, Y) \to \mathfrak{C}^{r-k}(X, J^k(X, Y))$$

the k-jet extension, where $\mathfrak{C}^r(X, Y)$ has the C^r topology of compact convergence (a Baire space). Let W be a C^{r-k} manifold (without boundary), $F \in \mathfrak{C}^{r-k}(W, J^k(X, Y))$, and

$$\mathfrak{C}^r_F(X, Y) = \{f \in \mathfrak{C}^r(X, Y) : j^k f \pitchfork F\}.$$

Finally, suppose W, X and Y are finite dimensional.

Jet transversality theorem (Thom). *If*

$$r > \max\{\dim X + \dim W - \dim J^k(X, Y), 0\},$$

then the subspace $\mathfrak{C}^r_F(X, Y) \subset \mathfrak{C}^r(X, Y)$ is residual for every C^{r-k} mapping $F: W \to J^k(X, Y)$.

Proof. First suppose X is compact. Then it is known that $\mathfrak{A} = j^k[\mathfrak{C}^r(X, Y)]$ has a natural structure of C^{r-k} manifold of mappings compatible with the topology of compact convergence [1]. Furthermore a standard computation using a local chart and a C^r characteristic function shows that the differential of the evaluation mapping

$$\mathrm{ev}: \mathfrak{A} \times X \to J^k(X, Y): (j^k f, x) \to j^k f(x)$$

is surjective at any point $(j^k f, x) \in \mathfrak{A} \times X$, so the evaluation map is transversal to any mapping $F: W \to J^k(X, Y)$. Thus if

$r > \max\{\dim X + \dim W - \dim J^k(X, Y), 0\}$, the Openness and Density Theorems 2.4 and 2.7 imply that $\mathcal{A}_{X,F} \subset \mathcal{A}$ is open and dense. Now suppose X is not compact. Then, using a countable covering of X by compact manifolds with boundary, a simple point set argument, and the proof above, $\mathcal{A}_{X,F} \subset \mathcal{A}$ is seen to be residual. But $j^k: \mathcal{C}^r(X, Y) \to \mathcal{A}$ is a homeomorphism so $\mathcal{C}^r_F(X, Y) = j^{k^{-1}}(\mathcal{A}_{X,F})$ is residual in $\mathcal{C}^r(X, Y)$.

References

1. R. Abraham, *Lectures of Smale on differential topology*, Columbia University, New York, 1962.
2. J. Eells, *On the geometry of function spaces*, Symposium Internacional de Topologiá Algebraica, pp. 303–308, Universidad Nacional Autónoma de México and UNESCO, Mexico City, 1958.
3. S. Lang, *Introduction to differentiable manifolds*, Interscience, New York, 1962.
4. H. I. Levine, *Singularities of differentiable mappings*. I, Mathematisches Institut der Universität Bonn, Bonn, 1959.
5. R. Palais, *From letters of R. Palais*, Columbia University, New York, 1962.
6. A. Sard, *The measure of the critical values of differentiable maps*, Bull. Amer. Math. Soc. **48** (1942), 883–890.
7. R. Thom, *Un lemme sur les applications differentiables*, Bol. Soc. Mat. Mexicana (2) **1** (1956), 59–71.
8. H. Whitney, *Differentiable manifolds*, Ann. of Math. **37** (1936), 645–680.

Columbia University