

This follows from the continuity of F , as before. The proof of the theorem is complete.

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TRANSVERSALITY IN MANIFOLDS OF MAPPINGS¹

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1. **Introduction.** Let X and Y be differentiable manifolds and \mathcal{A} a space of mappings from X to Y . A common problem in differential topology is to approximate a mapping in \mathcal{A} by another in \mathcal{A} which is transversal to a given submanifold $W \subset Y$. Thus if $\mathcal{A}_{X,W}$ is the subspace of mappings transversal to W it is important to know if $\mathcal{A}_{X,W}$ is dense in \mathcal{A} . Some famous examples are the Whitney immersion and embedding theorems [8] and the Thom transversality theorem [4; 7]. In the next section we give sufficient conditions for density in case \mathcal{A} is a Banach manifold. The proof of the density theorem is indicated in the third section, and in the final section the Thom transversality theorem is obtained as a corollary.

2. **Density theorems.** Throughout this section X will be a manifold with boundary, Y and Z manifolds, $W \subset Y$ a submanifold (W , Y , Z without boundary) all of class C^r , $r \geq 1$, and modelled on Banach spaces (see [3] for definitions).

2.1. **DEFINITION.** A C^r mapping $f: X \rightarrow Y$ is *transversal to W at a point $x \in X$* iff either $f(x) \notin W$, or $f(x) = w \in W$ and there exists a neighborhood U of $x \in X$ and a local chart (V, ψ) at $w \in Y$ such that

$$\psi: V \rightarrow E \times F: V \cap W \rightarrow E \times 0,$$

$\pi_1 \circ \psi$ is a diffeomorphism of $V \cap W$ onto an open set of E , and $\pi_2 \circ \psi \circ f|_U$ is a submersion [3, p. 20], where $\pi_1: E \times F \rightarrow E$ and

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$\pi_2: E \times F \rightarrow F$ are the projections. The mapping f is *transversal to W on a subset $K \subset X$* , $f|_K \pitchfork W$, iff f is transversal to W at every point $x \in K$; and f is *transversal to W* , $f \pitchfork W$, iff $f|_X \pitchfork W$.

For some basic properties of transversality see Lang [3, p. 22]. Suppose $f: X \rightarrow Y$ and $g: Z \rightarrow Y$ are C^r mappings, and let $\Delta \subset Y \times Y$ denote the diagonal.

2.2. DEFINITION. The mappings $f: X \rightarrow Y$ and $g: Z \rightarrow Y$ are *transversal at points $x \in X$ and $z \in Z$* iff the product $f \times g: X \times Z \rightarrow Y \times Y$ is transversal to Δ at (x, z) in the sense of Definition 2.1. The mappings are *transversal on sets $K \subset X$ and $M \subset Z$* , $f|_K \pitchfork g|_M$, iff

$$f \times g|_K \times M \pitchfork \Delta,$$

and they are *transversal*, $f \pitchfork g$, iff $f \times g \pitchfork \Delta$.

2.3. DEFINITION. A C^r manifold of mappings from X to Y is a set \mathcal{Q} of C^r mappings from X to Y which is a C^r manifold such that the evaluation mapping

$$\text{ev}: \mathcal{Q} \times X \rightarrow Y: (f, x) \rightarrow f(x)$$

is of class C^r .

For example it is known that if X is compact then the set $\mathcal{C}^r(X, Y)$ of all C^r mappings from X to Y has a natural structure of C^r manifold of mappings [1, p. 31; 2; 5].

If $K \subset X$ is any subset, and \mathcal{Q} a manifold of mappings from X to Y , let $\mathcal{Q}_{K,W} \subset \mathcal{Q}$ be the subspace of mappings which are transversal to W on the set K . The following is easy to prove.

2.4. OPENNESS THEOREM. *If $K \subset X$ is a compact set, $W \subset Y$ a closed submanifold, and $\mathcal{Q} \subset \mathcal{C}^r(X, Y)$ a C^r manifold of mappings, then the subset $\mathcal{Q}_{K,W}$ is open in \mathcal{Q} .*

Recall that a *residual set* in a topological space is a countable intersection of open dense sets, a *Baire space* is one in which every residual set is dense, and by the Baire category theorem every Banach manifold is a Baire space.

2.5. DENSITY THEOREM. *Let X be an n -manifold with boundary, $K \subset X$ any subset, Y a Banach manifold (without boundary), and $W \subset Y$ a closed submanifold (without boundary) of finite codimension q , all of class C^r . Let $\mathcal{Q} \subset \mathcal{C}^r(X, Y)$ be a C^r manifold of mappings. If the evaluation map of \mathcal{Q}*

$$\text{ev}: \mathcal{Q} \times X \rightarrow Y: (f, x) \rightarrow f(x)$$

is transversal to W on K and $r > \max(n - q, 0)$, then $\mathcal{Q}_{K,W} \subset \mathcal{Q}$ is residual.

This theorem is the main result, and may be generalized in several ways. For example suppose X, Y and Z are finite dimensional, $\mathcal{A} \subset C^r(X, Y)$ and $\mathcal{B} \subset C^r(Z, Y)$ are C^r manifolds of mappings, and $ev_{\mathcal{A}}$ and $ev_{\mathcal{B}}$ are the respective evaluation mappings. If $K \subset X$ and $M \subset Z$, let $\mathcal{A} \times \mathcal{B}_{K \times M} = \{(f, g) \in \mathcal{A} \times \mathcal{B} : f|_K \# g|_M\}$. Then the following are immediate.

2.6. COROLLARY. *If $ev_{\mathcal{A}}|_{\mathcal{A} \times K} \# ev_{\mathcal{B}}|_{\mathcal{B} \times M}$ and $r > \max(\dim X + \dim Z - \dim Y, 0)$, then $\mathcal{A} \times \mathcal{B}_{K \times M} \subset \mathcal{A} \times \mathcal{B}$ is residual.*

Let $\mathcal{B} = \{g\}$, a single map, and

$$\mathcal{A}_{K, g} = \{f \in \mathcal{A} : f|_K \# g\}.$$

2.7. COROLLARY. *If $ev_{\mathcal{A}}|_{\mathcal{A} \times K} \# g$ and*

$$r > \max(\dim X + \dim Z - \dim Y, 0),$$

then $\mathcal{A}_{K, g} \subset \mathcal{A}$ is residual.

If in 2.7 g is an embedding and W its image, then 2.5 is obtained with the condition “ W closed” deleted.

Note the symmetry of 2.2 and 2.6. Both may be extended to n -tuples of mappings having a common target, and the symmetry can be completed by allowing all sources to be manifolds with boundary, making only trivial modifications.

3. **Proof of the density theorem.** The Density Theorem 2.5 is proved from the following lemma by an easy point set argument.

DENSITY LEMMA. *If X has finite dimension n , $W \subset Y$ is a closed submanifold having finite codimension q , $\mathcal{A} \subset C^r(X, Y)$ is a C^r manifold of mappings with $r > \max(n - q, 0)$, and the evaluation mapping is transversal to W at a point $(f, x) \in \mathcal{A} \times X$, then there exists a neighborhood \mathcal{U} of $f \in \mathcal{A}$ and a neighborhood V of $x \in X$ such that $\mathcal{U}_V, w \subset \mathcal{U}$ is dense.*

If $f(x) \notin W$ the lemma is trivial. If $f(x) = w \in W$ the proof is immediate from these three propositions.

PROPOSITION A. *If the evaluation map is transversal to W at (f, x) and $f(x) = w \in W$, there are neighborhoods \mathcal{U} of $f \in \mathcal{A}$ and V of $x \in X$ such that every point $g \in \mathcal{U}$ is contained in a p -dimensional submanifold $\sum_{\sigma}^p, 0 \leq p \leq q$, such that $ev|_{\sum_{\sigma}^p \times V} \# W$.*

The proof of this proposition is prosaic, relying on techniques which have become standard since the publication of Lang’s book [3].

For the second proposition suppose \sum^p is a p -dimensional sub-

manifold of \mathcal{Q} , V an open set of X , and $\xi = \text{ev}| \sum^p \times V$ is transversal to W . Then $W' = \xi^{-1}(W)$ is a submanifold of codimension q of $\sum^p \times V$. Let $\sigma: W' \rightarrow \sum^p$ denote the restriction to W' of the projection $\sum^p \times V \rightarrow \sum^p$.

PROPOSITION B. *If σ is transversal to a point $f \in \sum^p$, then f is transversal to W on V .*

The proof of this proposition is a straightforward interpretation of the definitions.

The third proposition is a well-known theorem of Sard [6]. Recall that if $f: X \rightarrow Y$ is any C^1 mapping, a point $y \in Y$ is a *critical value* of f iff it is false that $f \pitchfork \{y\}$. Let Ω_f be the set of all critical values of f .

PROPOSITION C (SARD). *If $f: \mathbb{R}^s \rightarrow \mathbb{R}^t$ is of class C^r with $r > \max(s-t, 0)$, then $\Omega_f \subset \mathbb{R}^t$ has outer measure zero.*

4. The Thom transversality theorem. Let X be a C^r manifold with boundary, Y a C^r manifold (without boundary), $\pi^k: J^k(X, Y) \rightarrow X \times Y$ the k -jet bundle of C^k maps from X to Y , $0 \leq k \leq r$, and

$$j^k: \mathcal{C}^r(X, Y) \rightarrow \mathcal{C}^{r-k}(X, J^k(X, Y))$$

the k -jet extension, where $\mathcal{C}^r(X, Y)$ has the C^r topology of compact convergence (a Baire space). Let W be a C^{r-k} manifold (without boundary), $F \in \mathcal{C}^{r-k}(W, J^k(X, Y))$, and

$$\mathcal{C}_F^r(X, Y) = \{f \in \mathcal{C}^r(X, Y) : j^k f \pitchfork F\}.$$

Finally, suppose W , X and Y are finite dimensional.

JET TRANSVERSALITY THEOREM (THOM). *If*

$$r > \max\{\dim X + \dim W - \dim J^k(X, Y), 0\},$$

then the subspace $\mathcal{C}_F^r(X, Y) \subset \mathcal{C}^r(X, Y)$ is residual for every C^{r-k} mapping $F: W \rightarrow J^k(X, Y)$.

PROOF. First suppose X is compact. Then it is known that $\mathcal{Q} = j^k[\mathcal{C}^r(X, Y)]$ has a natural structure of C^{r-k} manifold of mappings compatible with the topology of compact convergence [1]. Furthermore a standard computation using a local chart and a C^r characteristic function shows that the differential of the evaluation mapping

$$\text{ev}: \mathcal{Q} \times X \rightarrow J^k(X, Y): (j^k f, x) \rightarrow j^k f(x)$$

is surjective at any point $(j^k f, x) \in \mathcal{Q} \times X$, so the evaluation map is transversal to any mapping $F: W \rightarrow J^k(X, Y)$. Thus if

$r > \max\{\dim X + \dim W - \dim J^k(X, Y), 0\}$, the Openness and Density Theorems 2.4 and 2.7 imply that $\mathcal{Q}_{X,F} \subset \mathcal{Q}$ is open and dense. Now suppose X is not compact. Then, using a countable covering of X by compact manifolds with boundary, a simple point set argument, and the proof above, $\mathcal{Q}_{X,F} \subset \mathcal{Q}$ is seen to be residual. But $j^k: \mathcal{C}^r(X, Y) \rightarrow \mathcal{Q}$ is a homeomorphism so $\mathcal{C}_F^r(X, Y) = j^{k^{-1}}(\mathcal{Q}_{X,F})$ is residual in $\mathcal{C}^r(X, Y)$.

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