LIPSCHITZ CLASSES OF FUNCTIONS AND DISTRIBUTIONS IN E_n

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Communicated by A. Zygmund, February 11, 1963

The results summarized here are the principle results of the author's doctoral dissertation presented at the University of Chicago and written under the direction of E. M. Stein. These results will appear soon with proofs.

We consider properties of classes of functions and distributions which are characterized by various smoothness and differentiability conditions. Similar classes have been studied by many investigators in recent years. Those papers touching most closely on the results given here are cited throughout this announcement. Special attention, however, is directed to the thorough list of references given recently by Nikolskiĭ in [9].

Our methods are analogous to those of Hardy and Littlewood in their study of analytic and harmonic functions in the unit disc (see [6]) extended to the n-dimensional nonperiodic case by consideration of "Poisson integrals" of tempered distributions in the same spirit as Stein and Weiss [11], and Stein [10].

We shall consistently denote by x and h elements of E_n, and by y elements of the positive real axis. By $L_p(E_n)=L_p$ we mean the normed linear space of measurable functions $f(x)$ for which the norm

$$\|f(x)\|_p = \|f\|_p = \left[\int_{E_n} |f(x)|^p dx\right]^{1/p}$$

is finite ($1 \leqq p < \infty$). Using

$$\|f(x)\|_\infty = \operatorname*{ess\,sup}_{x\in E_n} |f(x)|,$$

we define $L_\infty(E_n)=L_\infty$ analogously. We also need notation for some mixed norms. Suppose $f(x, h)$ is measurable in x and h. Then define

$$\|f(x, h)\|_{pq} = \left[\int_{h\in E_n} \|f(x, h)\|_p^q |h|^{-n} dh\right]^{1/q}, \qquad 1 \leqq q < \infty,$$

and

[1] Preparation of this report supported in part by the U. S. Army Research Office (Durham), Contract No. DA-31-124-ARO(D)-58.

$$\|f(x,h)\|_{p\infty} = \operatorname*{ess\,sup}_{h\in E_n} \|f(x,h)\|_p.$$

If $f(x, y)$ is measurable in x and y define

$$\|f(x,y)\|_{pq} = \left[\int_0^\infty \|f(x,y)\|_p^q y^{-1} dy\right]^{1/q}, \qquad 1 \leqq q < \infty,$$

$$\|f(x,y)\|_{pq}^* = \left[\int_0^1 \|f(x,y)\|_p^q y^{-1} dy\right]^{1/q}, \qquad 1 \leqq q < \infty,$$

$$\|f(x,y)\|_{p\infty} = \operatorname*{ess\,sup}_{y>0} \|f(x,y)\|_p,$$

and

$$\|f(x,y)\|_{p\infty}^* = \operatorname*{ess\,sup}_{0<y<1} \|f(x,y)\|_p.$$

If f is a tempered distribution let $J^\alpha f$ be the Bessel potential of order α (α real) of f. ($(J^\alpha f)^\frown = (1+4\pi^2|x|^2)^{-\alpha/2}\hat{f}$, $\hat{f}$ the Fourier transform of f.) (See Aronszajn and Smith [1], and Calderon [3].) Define $L_{p\alpha}$ as the normed linear Lebesgue space of tempered distributions f such that $f = J^\alpha\phi$, where $\phi \in L_p$, with norm $\|f\|_{p\alpha} = \|\phi\|_p$.

If $f \in L_{p\alpha}$ for some real α and $1 \leqq p \leqq \infty$, then let $f(x, y)$ be the Poisson integral of f. (See Stein and Weiss [11, pp. 35–45].)

Definition. Let α be a real number and α^* be the smallest nonnegative integer greater than α and α_* the largest nonpositive integer less than α. If $f(x, y)$ is the Poisson integral of a tempered distribution we will let $f^{(k)}(x, y)$ denote the kth derivative of $f(x, y)$ with respect to y, k a non-negative integer. We then define a normed linear Lipschitz space of tempered distributions $\Lambda(\alpha; p, q; E_n) = \Lambda(\alpha; p, q)$, α real, $1 \leqq p, q \leqq \infty$ as the set of tempered distributions $f \in L_{p\alpha_*}$ for which the norm $\|f\|_{\alpha;p,q} = \|f\|_{p,\alpha_*} + \|y^{\alpha^*-\alpha} f^{(\alpha)^*}(x,y)\|_{pq}^*$ is finite.

Theorem 1. *J^β is a topological isomorphism of $\Lambda(\alpha; p, q)$ onto $\Lambda(\alpha+\beta; p, q)$ for all α, β real, $1 \leqq p, q \leqq \infty$.*

For $0 < \alpha < 1$ the spaces $\Lambda(\alpha; p, \infty)$ are the usual Lipschitz spaces Lip(α, p) or in the notation of Zygmund [13, vol. I, pp. 43–45] Λ_α^p. $\Lambda(1; p, \infty)$ is the Λ_*^p space of Zygmund [13] or the Λ_1^p space of Calderon. More generally, our notation agrees with Calderon in that $\Lambda(\alpha; p, \infty)$ are Calderon's spaces Λ_α^p for all real α and $1 \leqq p \leqq \infty$ (see [3]). For each $q < \infty$, $\Lambda(\alpha; p, q)$ is a subspace of the so-called little Lipschitz spaces λ_α^p of [13]. We also show that $\Lambda(\alpha; p, p)$ is the W_p^α space of Soboleff for nonintegral $\alpha > 0$ and is the Besov space B_p^α for all $\alpha \geqq 0$. For more historical details and connections with the spaces of Nikolskiĭ, Slobedetsky, Solonnikov, Gagliardo, Aronszajn and others see Nikolskiĭ's recent survey [9] and the introduction to [12]. The precise connections are spelled out in the theorems below together with the fact that W_p^α is $L_{p\alpha}$ for non-negative integers α and $1 < p < \infty$.

THEOREM 2. *Let $f(x) \in L_p$, $1 \leqq p \leqq \infty$, and $f(x, y)$ be its Poisson integral. Define for a fixed q, $1 \leqq q \leqq \infty$:*

$$A = \left\| |h|^{-\alpha}[f(x+h) - f(x)] \right\|_{pq},$$

$$B = \left\| |h|^{-\alpha}[f(x+h) - 2f(x) + f(x-h)] \right\|_{pq},$$

$$C = \left\| y^{1-\alpha} f^{(1)}(x, y) \right\|_{pq},$$

$$D = \left\| y^{2-\alpha} f^{(2)}(x, y) \right\|_{pq},$$

$$\omega_1(t; p) = \sup_{0<|h|\leqq t} \left\| f(x+h) - f(x) \right\|_p,$$

$$\omega_2(t; p) = \sup_{0<|h|\leqq t} \left\| f(x+h) - 2f(x) + f(x-h) \right\|_p,$$

$$E = \left(\int_0^\infty [t^{-\alpha}\omega_1(t; p)]^q t^{-1} dt \right)^{1/q} \qquad \text{if } 1 \leqq q < \infty,$$

$$E = \sup_{t>0} t^{-\alpha}\omega_1(t; p) \qquad \text{if } q = \infty,$$

$$F = \left(\int_0^\infty [t^{-\alpha}\omega_2(t; p)]^q t^{-1} dt \right)^{1/q} \qquad \text{if } 1 \leqq q < \infty,$$

$$F = \sup_{t>0} t^{-\alpha}\omega_2(t; p) \qquad \text{if } q = \infty,$$

$$G = \left\| y^{-\alpha}[f(x, y) - f(x)] \right\|_{pq}.$$

If $0 < \alpha < 1$ and any of A, B, C, D, E, F, or G is finite, then so are all the others and the ratio of any pair is bounded above independent of $\|f\|_p$.

If $0 < \alpha < 2$ and any of B, D, or F is finite, then so are the other two and the ratio of any pair is bounded above independent of $\|f\|_p$.

DEFINITION. Let $r = (r_1, r_2, \cdots, r_n)$ be an n-vector with non-negative integer components. Define $|r| = r_1 + r_2 + \cdots + r_n$. If f is a tempered distribution we define $D^r f$ to be weak or distribution derivative $(\partial/\partial x_1)^{r_1}(\partial/\partial x_2)^{r_2} \cdots (\partial/\partial x_n)^{r_n} f$.

THEOREM 3. *Any pair of the following norms for $\Lambda(\alpha; p, q)$ are equivalent, α real, $1 \leqq p$, $q \leqq \infty$, where k, l, and s are non-negative integers and β, γ, δ, and ϵ are real numbers.*

(a) $\left\| y^{k-\alpha} f^{(k)}(x, y) \right\|_{pq} + \|f\|_{p\beta}$, $k > \alpha > 0$, $\beta < \alpha$.

(b) $\|y^{l-\alpha}f^{(l)}(x, y)\|_{pq}^{*}+\|f\|_{p\gamma}$, $l>\alpha$, $\gamma<\alpha$.
(c) $\sum_{|r|=s}\|D^{r}f\|_{\alpha-s;\ p,q}+\|f\|_{p\delta}$, $\delta<\alpha$.
(d) $\|\phi\|_{\alpha-\epsilon;p,q}$ *where* $f=J^{\epsilon}\phi$.

THEOREM 4. $\Lambda(\alpha_1; p_1, q_1)$ *is contained algebraically and topologically in* $\Lambda(\alpha_2; p_2, q_2)$ *if and only if*: $p_1\leqq p_2$, $\alpha_1-n/p_1\geqq\alpha_2-n/p_2$, *and when* $\alpha_1-n/p_1=\alpha_2-n/p_2$ *if* $q_1\leqq q_2$.

Let $\mathfrak{D}$ be the set of indefinitely differentiable functions on E_n with compact support, $\mathfrak{C}_0$ the set of continuous functions which vanish at "infinity," and $\mathfrak{C}_{0\beta}=\{f: f=J^{\beta}\phi,\ \phi\in\mathfrak{C}_0\}$.

THEOREM 5. (a) $\Lambda(\alpha; p, q)$, α *real*, $1\leqq p, q\leqq\infty$ *is complete.*
(b) $\mathfrak{D}$ *is dense in* $\Lambda(\alpha; p, q)$ *for* α *real*, $1\leqq p, q<\infty$, *and is also dense in* $\Lambda(\alpha;\ \infty, q)\cap\mathfrak{C}_{0\alpha_*}$ *for all real* α *and* $1\leqq q<\infty$.

The "if" part of parts (a) and (b) of the following theorem extends a result of Hirschman's (see [7]).

THEOREM 6. *Let* α *be real.*
If $1<p<\infty$
(a) $L_{p\alpha}\subset\Lambda(\alpha; p, q)$ *if and only if* $\max[p, 2]\leqq q\leqq\infty$,
(b) $\Lambda(\alpha; p, q)\subset L_{p\alpha}$ *if and only if* $1\leqq q\leqq\min[p, 2]$.
If $p=1$ *or* ∞
(c) $L_{p\alpha}\subset\Lambda(\alpha; p, q)$ *if and only if* $q=\infty$,
(d) $\Lambda(\alpha; p, q)\subset L_{p\alpha}$ *if and only if* $q=1$.
When the inclusion is valid, the inclusion map is continuous.

NOTE. From Theorem 6 it follows that $\Lambda(\alpha;\ 2, 2)=L_{2\alpha}$, which result was obtained by Aronszajn and Smith [1].

Let R be a subspace of E_n. If $f(x)$ is a function defined on E_n and the Poisson integral $f(x, y)$ of $f(x)$ converges in some L_p norm $(1\leqq p\leqq\infty)$ on R to a function $g(x)$, then we define $f|_R(x)=g(x)$ for $x\in R$. It is clear that such a function is unique up to a set of measure zero in R. Suppose an image of E_k $(0\leqq k\leqq n)$ is embedded as a linear subspace of E_n. We call this subspace again E_k.

THEOREM 7. (a) *If* $f(x)\in\Lambda(\alpha; p, q; E_n)$, $\beta=\alpha-n/p+k/r>0$, $r\geqq p$, *then* $f|_{E_k}(x)$ *is defined and is in* $\Lambda(\beta; r, q; E_k)$.
(b) *If* $f\in\Lambda(\alpha; p, q; E_k)$, $\beta=\alpha-k/p+n/r>0$, $r\geqq p$, *then there exists* $g(x)\in\Lambda(\beta; r, q; E_n)$ *such that* $g|_{E_k}(x)=f(x)$.
The restriction and embedding maps thus defined are continuous.

For results of a similar nature see [1] and [9].

Theorems 1–7 also hold in the periodic case with appropriate modifications resulting from the compactness of the fundamental torus,

T_n. Thus in Theorem 5 the extra conditions imposed on the $p=\infty$ case may be dropped. Theorem 4 for the periodic case takes the form:

THEOREM 4'. *$\Lambda(\alpha_1;\ p_1,\ q_1;\ T_n)$ is contained algebraically and topologically in $\Lambda(\alpha_2;\ p_2,\ q_2;\ T_n)$ α_1, α_2 real, $1\leqq p_1, p_2, q_1, q_2\leqq\infty$ if and only if $\alpha_1\geqq\alpha_2$, $\alpha_1-n/p_1\geqq\alpha_2-n/p_2$, and when $\alpha_1-n/p_1=\alpha_2-n/p_2$ or $\alpha_1=\alpha_2$ if $q_1\leqq q_2$.*

In the periodic case we consider functions and distributions defined on the "fundamental torus" T_n of E_n, that is, the set of points $x\in E_n$ such that $-1/2<x_i\leqq 1/2$, $x=(x_1, x_2, \cdots, x_n)$.

Let $k=(k_1, \cdots, k_n)$ be a lattice point in E_n (i.e., an n-vector with integer components). $f(x)$ is extended to a periodic function in E_n by defining $f(x+k)=f(x)$ for all $x\in E_n$ and all lattice points $k\in E_n$. The theorems mentioned above then extend to the periodic case by "periodizing" the kernels defining the Poisson integral and Bessel potential of a function. (See Calderon and Zygmund [5] for a discussion of this approach of proving periodic versions of a theorem by "periodizing" the results for the general case.)

We state now a theorem whose main interest lies in the case $p=1$, ∞, the result for $1<p<\infty$ following from well-known results that singular integral operators of the type to be described are bounded operators from L_p into itself for $1<p<\infty$. (See [5].)

Let $x'=x/|x|$ for $x\in E_n$. $K(x)=\Omega(x')/|x|^n$ where $\int_\Sigma \Omega(x')dx'=0$ (Σ is the unit of sphere of E_n), and letting $\omega(t)$ be the rectified modulus of continuity of $\Omega(x')$ on Σ (i.e., $\omega(t)=\sup_{0<|s-r|\leqq t;\, s,r\in\Sigma}|\Omega(s)-\Omega(r)|$) we suppose that $\int_0^1\omega(t)t^{-1}dt$ is finite.

Let $K^*(x)$ be the peroidization of $K(x)$ in the sense of [5] and define

$$K_\epsilon^*(x) = \begin{cases} K^*(x) & \text{if } x \geqq \epsilon, \\ 0 & \text{if } x < \epsilon, \end{cases}$$

$\tilde{f}(x)=\lim_{\epsilon\to 0}\int_{T^n} f(x-h)K_\epsilon^*(h)dh$, when it exists.

It is well known that for $f(x)\in L_p(T_n)$, $1\leqq p\leqq\infty$, that $\tilde{f}(x)$ exists a.e.

THEOREM 8. *Define $Tf=\tilde{f}$. T maps $\Lambda(\alpha;\ p,\ q;\ T_n)$ continuously into itself, α real, $1\leqq p$, $q\leqq\infty$.*

The case $\omega(t)=O(t^\beta)$ for $\beta>\alpha$ was treated by Calderon and Zygmund in [5, p. 262].

The operators T so defined are translation invariant operators. We may continue the analysis begun by Zygmund in [14], and first show that if T is a linear translation invariant operator mapping some Λ space into itself, that restricted to $\mathfrak{D}$, T is represented by convolution with a unique tempered distribution A_T. Thus, if $\phi \in \mathfrak{D}$, then $T\phi = A_T * \phi$. We then show:

THEOREM 9. *If T is a linear translation invariant operator mapping $\Lambda(\alpha; \infty, \infty)$ or $\Lambda(\alpha; 1, \infty)$, for some real α, continuously into itself, then $A_T \in \Lambda(0; 1, \infty)$. If A is a tempered distribution and $A \in \Lambda(0; 1, \infty)$, then $T\phi = A * \phi$, $\phi \in \mathfrak{D}$, is an operator which extends uniquely to the closure of $\mathfrak{D}$ in $\Lambda(\alpha; p, q)$ (see Theorem 5b) and which maps such $\Lambda(\alpha; p, q)$ continuously into itself.*

The following result is a consequence of a theorem of Lions in [8] on the duals of certain "trace spaces."

THEOREM 10. *Let α be real, $1 < p, q < \infty$, $1/p + 1/p' = 1$, $1/q + 1/q' = 1$. Then the topological dual of $\Lambda(\alpha; p, q)$ is identifiable, topologically and algebraically, with $\Lambda(-\alpha; p', q')$. It follows that $\Lambda(\alpha; p, q)$ is reflexive provided $1 < p, q < \infty$.*

When Theorem 10 is combined with results of Calderon [4] on intermediate spaces, we obtain the following interpolation theorem of the Riesz-Thorin type.

THEOREM 11. *Suppose $1 < p_i, q_i, r_i, s_i < \infty$; α_i, β_i real; $i = 0, 1$. Then if T is a linear operator defined on $\mathfrak{D}$ with values in the space of tempered distributions and*

$$\|T\phi\|_{\beta_i; r_i, s_i} \leqq M_i \|\phi\|_{\alpha_i; p_i q_i}, \qquad i = 0, 1, \phi \in \mathfrak{D},$$

M_i independent of ϕ, then for each $0 \leqq t \leqq 1$, $\phi \in \mathfrak{D}$,

$$(\#) \qquad \|T\phi\|_{\beta; r, s} \leqq M_0^{1-t} M_1^t \|\phi\|_{\alpha; p, q}$$

where $\alpha = (1-t)\alpha_0 + t\alpha_1$; $\beta = (1-t)\beta_0 + t\beta_1$; $1/p = (1-t)/p_0 + t/p_1$; $1/q = (1-t)/q_0 + t/q_1$; $1/r = (1-t)/r_0 + t/r_1$; $1/s = (1-t)/s_0 + t/s_1$ and the operator has unique extension to all of $\Lambda(\alpha; p, q)$ preserving (#).

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CORRECTION TO *A POLYNOMIAL ANALOG OF THE GOLDBACH CONJECTURE*[1]

BY DAVID HAYES

Communicated by E. E. Moise, April 20, 1963

On page 116 of this paper, I state that if $r<2h$, then $\pi_K(r, d)\leqq d$ for $d>1$. This will be true in general only when H is an irreducible. However, the proof will still go through if either (1) H is square-free or else (2) $h+1$ is not divisible by the characteristic of the underlying finite field. That one of these conditions hold should therefore be added to Theorem 2 as a hypothesis.

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[1] Bull. Amer. Math. Soc. **69** (1963), 115–116.