THE PERMANENT ANALOGUE OF THE HADAMARD DETERMINANT THEOREM

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1. **Statement of results.** In [2; 3] it was conjectured that if $A=(a_{ij})$ is an n-square positive semi-definite hermitian matrix then

$$\text{per } A \geqq \prod_{i=1}^{n} a_{ii}. \tag{1}$$

Here per A denotes the permanent of A: per $A = \sum_\sigma \prod_{i=1}^n a_{i\sigma(i)}$ where the summation is over the whole symmetric group of degree n. It was announced [1] and later proved [2] that per $(A) \geqq \det A$ and the Hadamard determinant theorem suggests that the product of the main diagonal entries of A in fact separates the permanent and the determinant of A. In this note we sketch a proof of an inequality that is substantially stronger than (1). Let $A(i)$ denote the principal submatrix of A obtained by deleting row and column i.

THEOREM. *If A is an $(r+1)$-square positive semi-definite hermitian matrix then*

$$(r+1)a_{11} \text{ per } A(1) \geqq \text{per } A \geqq a_{11} \text{ per } A(1). \tag{2}$$

If A has a zero row then (2) *is equality throughout. If A has no zero row then the lower equality holds if and only if $A = a_{11} \dotplus A(1)$; the upper equality holds if and only if A is of rank* 1.

We remark that what is true for $A(1)$ is true for any $A(i)$ because the permanent is unaltered by permutation of the rows and columns.

By an obvious induction on r we have the

COROLLARY. *If A is an n-square positive semi-definite hermitian matrix then*

$$\text{per } A \geqq \prod_{i=1}^{n} a_{ii} \tag{3}$$

with equality if and only if A has a zero row or A is a diagonal matrix.

2. **Proof of theorem.** We outline the proof of the theorem. Let U be an n-dimensional unitary space with inner product (x, y). For $1 \leqq r \leqq n$ define $U^{(r)}$ to be the space of r-tensors on U; that is, $U^{(r)}$ is the dual space of the space of all multilinear complex valued func-

tions of r-tuples of vectors from U. If $x_1, \cdots, x_r$ are in U then $x_1\otimes \cdots \otimes x_r \in U^{(r)}$ is defined by $x_1\otimes \cdots \otimes x_r(\phi)=\phi(x_1, \cdots, x_r)$ for any multilinear functional ϕ. Define the completely symmetric operator S: $U^{(r)}\rightarrow U^{(r)}$ by

$$S(x_1 \otimes \cdots \otimes x_r) = \frac{1}{r!} \sum_{\sigma} x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(r)}$$

where σ runs over the symmetric group of degree r. Then define the symmetric product of $x_1, \cdots, x_r$ by $x_1 \cdots x_r = S(x_1\otimes \cdots \otimes x_r)$. The range of S is the symmetry class of completely symmetric tensors denoted by $U_{(r)}$. It is well known that an inner product in $V_{(r)}$ is given by

$$(4) \qquad (x_1 \cdots x_r, y_1 \cdots y_r) = \frac{1}{r!} \operatorname{per}((x_i, y_j)), \qquad i,j = 1, \cdots, r.$$

Now let $G_{r,n}$ denote the totality of nondecreasing sequences of length r chosen from $1, \cdots, n$ and for $\omega \in G_{r,n}$ define $\mu(\omega)$ to be the product of the factorials of the multiplicities of the distinct integers in ω; e.g. $\mu(1, 1, 4, 4, 4, 5)=2!3!$. From (4) it follows that if $e_1, \cdots, e_n$ is an orthonormal (o.n.) basis in U then $\sqrt{(r!/\mu(\omega))}e_\omega$, $\omega \in G_{r,n}$, is an o.n. basis in $U_{(r)}$: here $e_\omega = e_{\omega_1} \cdots e_{\omega_r}$.

Now let e_1 be a unit vector in U and complete to an o.n. basis of U with $e_2, \cdots, e_n$. Next define a linear map T: $U_{(r)}\rightarrow U_{(r+1)}$ by $T(x_1 \cdots x_r) = e_1 \cdot x_1 \cdots x_r$. From (4) it follows that if no $x_i=0$, $i=1, \cdots, r$, then the Rayleigh quotient

$$(5) \qquad \frac{(Tx_1 \cdots x_r, Tx_1 \cdots x_r)}{(x_1 \cdots x_r, x_1 \cdots x_r)}$$

is given by

$$\frac{1}{r+1} \operatorname{per} A/\operatorname{per} A(1)$$

where $A = a_{ij}$ is the Gram matrix based on the ordered set $e_1, x_1, \cdots, x_r$: that is,

$$\begin{aligned} a_{i+1,j+1} &= (x_i, x_j), & 1 \leqq i, j \leqq r, \\ a_{11} &= (e_1, e_1), & \\ a_{1\,j+1} &= \bar{a}_{j+1\,1} = (e_1, x_j), & j = 1, \cdots, r. \end{aligned}$$

Let $H = T^*T$, then from (5) the problem becomes equivalent to determining the maximum and minimum eigenvalues of H and the corresponding eigenvectors. It turns out that a matrix representation of H with respect to the lexicographically ordered basis $\sqrt{(r!/\mu(\omega))}e_\omega$, $\omega \in G_{r,n}$, is diagonal and the eigenvalues of H are

$$\frac{\mu(1, \omega)}{\mu(\omega)} \cdot \frac{1}{r+1} \tag{6}$$

with corresponding eigenvectors e_ω: here $(1, \omega)$ designates the sequence $(1, \omega_1, \cdots, \omega_r) \in G_{r+1,n}$. The expression (6) is bounded below by $1/(r+1)$ and takes on this value if and only if $\omega_1 > 1$. This means that the lower inequality can hold in (2) if and only if $x_1 \cdots x_r$ is in the space spanned by e_ω, $\omega \in G_{r,n}$, $\omega_1 > 1$. We then prove that this happens if and only if $(x_i, e_1) = 0, i = 1, \cdots, r$, i.e., that $A = a_{11} \dotplus A(1)$. Similarly, the expression (6) is bounded above by 1 and takes on this value only for the sequence $\omega_i = 1$, $i = 1, \cdots, r$. This means that the upper equality can hold in (2) if and only if $x_1 \cdots x_r$ is a multiple of $e_1 \cdots e_1$. According to [2, Theorem 3] this happens only in case $x_i = d_i e_1$, $i = 1, \cdots, r$, and this leads to the condition that rank of A is 1.

References

1. Marvin Marcus and Morris Newman, *The permanent function as an inner product*, Bull. Amer. Math. Soc. **67** (1961), 223–224.

2. ———, *Inequalities for the permanent function*, Ann. of Math. (2) **75** (1962), 47–62.

3. Marvin Marcus and Henryk Minc, *The Pythagorean theorem in certain symmetry classes of tensors*, Trans. Amer. Math. Soc. **104** (1962), 510–515.

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