UNIQUENESS, STABILITY AND ERROR ESTIMATION

BY RAYMOND M. REDHEFFER

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As in the previous note [1] B is a bounded region, $x=(x_i)\in B$, and subscripts on u, v, w, c denote partial differentiation. We assume $u, v, w, c\in C^{(2)}$ in B and continuous in $\bar{B}$, although these conditions are unnecessarily restrictive.[1] The normal derivative u_ν is as in [1], $p=+$ or $p=-$, ϵ^p and δ^p are nonnegative constants, and $\|\ \|$ denotes the Euclidean norm. Conditions with T are for $x\in B$, those with R for $x\in\partial B$.

As an introduction to the problems we shall consider, let

$$Tu = a(x)u + \sum a_i(x)u_i - \sum a_{ij}(x)u_{ij}, \qquad Ru = u - k(x)u_\nu$$

where $a(x)\geqq A$, $|a_1(x)|\leqq A_1$, $a_{11}(x)=1$, $[a_{ij}(x)]\geqq 0$, $0\leqq k(x)\leqq K$, and $|x_1|\leqq b$. Suppose $y=y(x_1)$ satisfies, for $|x_1|\leqq b$:

$$E = \inf(Ay - A_1|y'| - y'') > 0, \qquad D = \inf(y - K|y'|) > 0.$$

Then the inequalities

$$p(Tu - Tv) \leqq \epsilon^p \quad\text{and}\quad p(Ru - Rv) \leqq \delta^p \tag{1}$$

imply $p(u-v)\leqq y(x_1)\max(\epsilon^p/E, \delta^p/D)$. The condition on a_1 can be replaced by $a_1\geqq -A_1$ if $y'\geqq 0$ and by $a_1\leqq A_1$ if $y'\leqq 0$.

The usefulness of this result and the ease of its proof suggest that it be extended to nonlinear problems, and that is our purpose. We first consider the operator and continuity conditions:

$$\begin{gathered} Tu = a(x, u, u_i) - \sum a_{ij}(x, u_k)u_{ij}, \\ \|a_{ij}(x, u_k) - a_{ij}(x, v_k)\| \leqq A_1\|u_k - v_k\|, \\ p[a(x, u, u_i) - a(x, v, v_i)] \geqq -A_1\|u_i - v_i\| \text{ for } p(u-v) > 0. \end{gathered} \tag{2}$$

Usually these hold only for u and v, but in Theorem 1 they hold for every pair of solutions, the constants depending on the pair:

THEOREM 1. *Let T be as in* (2) *and let $Ru=k_1(x, u)-k_2(x, u_\nu)$ where k_2 is nondecreasing and k_1 is strictly increasing in the last argument. Suppose the problem $Tw=\tau(x)$, $Rw=\rho(x)$ has solutions of the following two types:*

[1] The main condition really needed is that $p(u-v)$ be upper semicontinuous in the closure of the set where $p(u-v)>0$.

(i) *a solution* $w=u$ *such that* $[a_{ij}(x, u_k)] \geqq 0$, $x \in B$;
(ii) *a solution* $w=v$ *such that, for some function* $c(x)$,

$$\inf_x \sum a_{ij}(x, v_k)c_ic_j > 0, \qquad \inf_x \sum a_{ij}(x, v_k)c_{ij} > -\infty, \qquad x \in B.$$

Then $u=v$, *and there is no other solution.*

The hypothesis (i) can be written in the form

$$m_1(x)[a_{ij}(x, u_k)] + m_2(x)[a_{ij}(x, v_k)] \geqq 0 \tag{3a}$$

and the hypothesis (ii) in the form

$$\begin{gathered} \sum m_{ij}c_ic_j \geqq \mu_1, \qquad \sum m_{ij}c_{ij} \geqq \mu_2, \\ m_{ij} = m_3(x)a_{ij}(x, u_k) + m_4(x)a_{ij}(x, v_k). \end{gathered} \tag{3b}$$

We now suppose the m_i are any functions whatever satisfying

$$m_1 + m_2 = m_3 + m_4 = 1, \quad |m_i| \leqq M_i, \quad |m_1 - m_3| \leqq M_{13}. \tag{3c}$$

A hypothesis H is said to hold "in the neighborhood of the maximum" if to each fixed compact subset $S \subset B$ corresponds a positive constant, η, such that H holds at those points of S where simultaneously

$$\|u_i - v_i\| < \eta, \qquad m - \eta < p(u - v) < m, \qquad m = \sup p(u - v).$$

THEOREM 2. *Let* R *be as in Theorem* 1, *and let the following hypothesis hold in the neighborhood of the maximum*: T *has the form* (2), *and there exist multipliers* m_i *and a function* c *such that* (3) *holds with* $\mu_1 > 0$, $\mu_2 > -\infty$. *Then*

$$p(Tu - Tv) \leqq 0 \quad \textit{and} \quad p(Ru - Rv) \leqq 0 \Rightarrow p(u - v) \leqq 0.$$

This result (which contains Theorem 1) follows by setting $y = \alpha_0 - \alpha e^{\beta c}$ and considering the behavior of $p(u-v)-y$ at its maximum. A similar method yields stability when

$$Ru = u - k(x, u_\nu), \; p[k(x, u_\nu) - k(x, v_\nu)] \leqq K(|s|) \quad \text{for } p(u_\nu - v_\nu) \leqq s, \tag{4}$$

the function $K(s)$ being continuous, nonnegative, and increasing for $s \geqq 0$. Assuming

$$\|u_{ij}\| \leqq U_2, \quad \|v_{ij}\| \leqq V_2, \quad 0 \leqq c \leqq C, \quad \|c_i\| \leqq 1, \quad \|c_{ij}\| \leqq C_2 \tag{5}$$

we have:

THEOREM 3. *Let* (1)–(5) *hold with* $\mu_1 > 0$, *let* $\lambda > 0$, *and let* $\mu_1\beta = 1$ *or*

$$\mu_1\beta = 1 + A_1 + A_2(M_2U_2 + M_1V_2) + \lambda A_2M_{13} - \mu_2,$$

whichever is larger. Let $\epsilon^p \leqq \epsilon_0$, *where* $\epsilon_0(\beta + C_2) = \lambda$ sech βC, *and define* $\xi^p = \epsilon^p$ sinh βC. *Then* $p(u-v) \leqq \delta^p + C\xi^p + K(\xi^p)$.

If $(c_{ij}) \geqq 0$ we can replace C_2 and μ_2 by 0 in Theorem 3.
Similar results are valid for the operator

$$Tu = a(x, u, u_i) - \sum a_{ij}(x, u_k)u_{ij} - \sum a_{ijkl}(x, u_m)u_{ij}u_{kl} \tag{6a}$$

where u_m, like u_i and u_k, means $(u_1, u_2, \cdots, u_n)$. Denoting the function on the right of (6a) by $-f$, we set

$$\begin{aligned} f_{ij}(u) &= a_{ij}(x, u_k) + \sum [a_{ijkl}(x, u_m) + a_{klij}(x, u_m)]u_{kl}, \\ M_{ij} &= f_{ij}(u) + f_{ij}(v). \end{aligned} \tag{6b}$$

THEOREM 4. *Let T be as in* (6), *let* $Ru = k(x, u, u_\nu)$, *and let* α *and* β *be nonnegative constants. Suppose* $(M_{ij}) \geqq 0$ *for* $u_i = v_i$, *and further*:

$$\text{(i)}\ \ p[a(x, u, u_i) - a(x, v, v_i)] > \epsilon^p \qquad \textit{for } p(u-v) > \alpha, \quad u_i = v_i,$$

$$\text{(ii)}\ \ p[k(x, u, u_\nu) - k(x, v, v_\nu)] > \delta^p \qquad \textit{far } p(u-v) > \beta, \quad pu_\nu \leqq pv_\nu.$$

Then (1) *implies* $p(u-v) \leqq \max(\alpha, \beta)$.

The proof follows by consideration of $p(u-v) - y$, $y = \max(\alpha, \beta)$. We now assume

$$\|a_{ijkl}(x, u_m) - a_{ijkl}(x, v_m)\| \leqq B_2\|u_m - v_m\|. \tag{7}$$

The procedure used for Theorem 2 yields:

THEOREM 5. *Let T be as in* (6) *and R as in Theorem* 1. *Suppose the following hold in the neighborhood of the maximum*:

(i) *for some* $c(x)$ *the matrix* (M_{ij}) *satisfies*

$$(M_{ij}) \geqq 0, \qquad \inf_x \sum M_{ij}c_ic_j > 0, \qquad \inf_x \sum M_{ij}c_{ij} > -\infty,$$

(ii) *the continuity conditions* (2) *and* (7) *are valid.*
Then $p(Tu - Tv) \leqq 0$ *and* $p(Ru - Rv) \leqq 0 \Rightarrow p(u-v) \leqq 0$.

If the coefficients in (6) do not involve the first derivatives then (i) can be replaced by (i′): The matrix $(M_{ij}) \geqq 0$ and has only countably many zeros in B. This extension holds if, instead of u, $v \in C^{(2)}$, we have only existence of $(u-v)_i$ at each x for some $i = i(x)$. Theorem 5 implies a uniqueness theorem that strongly generalizes the results of Rellich [3].

As in [2] the Lipschitz-type terms $A\|u_i - v_i\|$ in Theorems 1, 2 and 5 can be replaced by $g(\|u_i - v_i\|)$ where the function $g(s)$ is positive, continuous and increasing for $s > 0$, and $\int_0^1 ds/g(s) = \infty$. By requiring

$c_\nu > 0$ and strict monotony of $k_2(x, u_\nu \uparrow)$ on part of ∂B one readily extends these results to mixed boundary-value problems. Corners are allowed, even if $Ru = -u_\nu$ at all but one point of ∂B.

Bibliography

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University of California, Los Angeles