

SINGULAR INTEGRALS AND PARABOLIC EQUATIONS

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Communicated by A. Zygmund, February 11, 1963

1. Introduction. A. P. Calderón and A. Zygmund [1; 2] have studied a class of singular integrals, proving that such integrals generate continuous linear transformations of L^p into L^p , $1 < p < \infty$. One of the many applications of their results is the derivation of integral estimates for derivatives of solutions of the Poisson equation $\Delta u = f$, where $\Delta = \partial^2/\partial x_1^2 + \dots + \partial^2/\partial x_n^2$. Corresponding results have been obtained for a different class of singular integrals; an application of these results is the derivation of integral estimates for derivatives of solutions of the parabolic equation $u_t - \Delta u = f$. We shall briefly outline the development of the singular integrals of Calderón and Zygmund as applied to the equation $\Delta u = f$, and then give the parallel development for the singular integrals associated with the equation $u_t - \Delta u = f$.

2. The equation $\Delta u = f$. In n -dimensional Euclidean space R^n let $\Gamma(x)$ be the fundamental solution of Laplace's equation,

$$\Gamma(x) = -\frac{1}{2\pi} \log \frac{1}{|x|}, \quad n = 2,$$

$$\Gamma(x) = \frac{1}{(2-n)\omega_n} |x|^{2-n}, \quad n > 2,$$

where $|x| = (x_1^2 + \dots + x_n^2)^{1/2}$ and ω_n is the area of the sphere $|x| = 1$. Let

$$(1) \quad u(x) = \int_{R^n} \Gamma(x-y)f(y)dy$$

where $f \in L^p(R^n)$. Then [1] the second partial derivatives of u exist almost everywhere, and

$$(2) \quad u_{x_i x_j} = \frac{1}{n} \delta_{ij} f(x) + \int_{R^n} k_{ij}(x-y)f(y)dy,$$

where $\delta_{ii} = 1$, $\delta_{ij} = 0$, $i \neq j$, and

$$k_{ij}(x) = \Gamma_{x_i x_j} = \frac{1}{\omega_n} |x|^{-n} \left(\delta_{ij} - n \frac{x_i x_j}{|x|^2} \right).$$

In particular, $\Delta u = f$. The kernel $k = k_{ij}$ has the properties that

$$(3) \quad k(\alpha x) = \alpha^{-n}k(x), \quad \alpha > 0,$$

$$(4) \quad \int_{|x|=1} k(x) d\sigma = 0.$$

Now if $k(x)$ is any function satisfying (3) and (4), together with a certain mild smoothness or boundedness condition [1; 2], then the mapping

$$(5) \quad \rightarrow \int_{R^n} k(x - y)f(y)dy \equiv \lim_{\epsilon \rightarrow 0+} \int_{|x-y|>\epsilon} k(x - y)f(y)dy$$

is a continuous transformation of L^p into L^p , for $1 < p < \infty$. In (2) the integral is also interpreted in the principal value sense of (5). The integral in (5) converges in L^p and pointwise almost everywhere. In particular, the derivatives $u_{x_i x_j}$ in (2) satisfy

$$\|u_{x_i x_j}\|_{L^p} \leq A_p \|f\|_{L^p}.$$

3. **The equation** $u_t - \Delta u = f$. For $0 < t < \infty$, $x \in R^n$, let $\Gamma(x, t)$ be the fundamental solution of the heat equation ($u_t - \Delta u = 0$),

$$\Gamma(x, t) = (4\pi t)^{-n/2} \exp\left(-\frac{|x|^2}{4t}\right).$$

Let

$$(1') \quad u(x, t) = \int_0^t \int_{R^n} \Gamma(x - y, t - s)f(y, s)dyds,$$

where $f \in L^p(R^n \times (0, \infty))$. The analogue of (2) is

$$(2') \quad \begin{aligned} u_{x_i x_j} &= \int_0^t \int_{R^n} k_{ij}(x - y, t - s)f(y, s)dyds, \\ u_t &= f(x, t) + \int_0^t \int_{R^n} k'(x - y, t - s)f(y, s)dyds. \end{aligned}$$

Here

$$\begin{aligned} k_{ij}(x, t) &= \Gamma_{x_i x_j} = (4\pi)^{-n/2} t^{-n/2-1} \left(-\frac{1}{2} \delta_{ij} + \frac{x_i x_j}{4t}\right) \exp\left(-\frac{|x|^2}{4t}\right), \\ k'(x, t) &= \Gamma_t = (4\pi)^{-n/2} t^{-n/2-1} \left(-\frac{1}{2} n + \frac{|x|^2}{4t}\right) \exp\left(-\frac{|x|^2}{4t}\right). \end{aligned}$$

In particular, $u_t - \Delta u = f$.

Now let us consider kernels $k(x, t)$ satisfying

$$(3') \quad \begin{aligned} k(x, t) &= 0, & t < 0, \\ k(\alpha x, \alpha^2 t) &= \alpha^{-n-2} k(x, t), & \alpha > 0, \end{aligned}$$

$$(4') \quad \int_{R^n} k(x, 1) dx = 0,$$

together with certain mild smoothness and boundedness conditions. For instance, it is sufficient to require that $|k(x, 1)| + |k_{x_i}(x, 1)| \leq a e^{-b|x|}$ for some $a > 0, b > 0$. All these properties are satisfied by the kernels $k_{ij}(x, t)$ and $k'(x, t)$. Note the analogy between conditions (3'), (4') and (3), (4), respectively.

If $k(x, t)$ is any such function, then we shall consider the mapping

$$(5') \quad \begin{aligned} f &\rightarrow \int_0^t \int_{R^n} k(x-y, t-s) f(y, s) dy ds \\ &\equiv \lim_{\epsilon \rightarrow 0+} \int_0^{t-\epsilon} \int_{R^n} k(x-y, t-s) f(y, s) dy ds, \end{aligned}$$

the integral converging in L^p . The integrals in (2') are also interpreted in this sense. Moreover the mapping (5') defines a continuous transformation of L^p into L^p , $1 < p < \infty$. Thus, for example, the derivatives $u_{x_i x_j}$ and u_t in (2') satisfy

$$\|u_{x_i x_j}\|_{L^p} + \|u_t\|_{L^p} \leq A_p \|f\|_{L^p}.$$

The precise statements and proofs of these and allied results will appear elsewhere. The proofs are similar to some of the arguments in [1].

REFERENCES

1. A. P. Calderón and A. Zygmund, *On the existence of certain singular integrals*, Acta Math. **88** (1952), 85-139.
2. ———, *On singular integrals*, Amer. J. Math. **78** (1956), 289-309.

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