SINGULAR INTEGRALS AND PARABOLIC EQUATIONS

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1. **Introduction.** A. P. Calderón and A. Zygmund [1; 2] have studied a class of singular integrals, proving that such integrals generate continuous linear transformations of L^p into L^p, $1<p<\infty$. One of the many applications of their results is the derivation of integral estimates for derivatives of solutions of the Poisson equation $\Delta u = f$, where $\Delta = \partial^2/\partial x_1^2 + \cdots + \partial^2/\partial x_n^2$. Corresponding results have been obtained for a different class of singular integrals; an application of these results is the derivation of integral estimates for derivatives of solutions of the parabolic equation $u_t - \Delta u = f$. We shall briefly outline the development of the singular integrals of Calderón and Zygmund as applied to the equation $\Delta u = f$, and then give the parallel development for the singular integrals associated with the equation $u_t - \Delta u = f$.

2. **The equation** $\Delta u = f$. In n-dimensional Euclidean space R^n let $\Gamma(x)$ be the fundamental solution of Laplace's equation,

$$\Gamma(x) = -\frac{1}{2\pi}\log\frac{1}{|x|}, \qquad n = 2,$$

$$\Gamma(x) = \frac{1}{(2-n)\,\omega_n}|x|^{2-n}, \qquad n > 2,$$

where $|x| = (x_1^2 + \cdots + x_n^2)^{1/2}$ and ω_n is the area of the sphere $|x| = 1$. Let

$$u(x) = \int_{R^n} \Gamma(x-y) f(y)\,dy \tag{1}$$

where $f \in L^p(R^n)$. Then [1] the second partial derivatives of u exist almost everywhere, and

$$u_{x_i x_j} = \frac{1}{n}\delta_{ij} f(x) + \int_{R^n} k_{ij}(x-y) f(y)\,dy, \tag{2}$$

where $\delta_{ii} = 1$, $\delta_{ij} = 0$, $i \neq j$, and

$$k_{ij}(x) = \Gamma_{x_i x_j} = \frac{1}{\omega_n}|x|^{-n}\left(\delta_{ij} - n\frac{x_i x_j}{|x|^2}\right).$$

In particular, $\Delta u = f$. The kernel $k = k_{ij}$ has the properties that

$$k(\alpha x) = \alpha^{-n}k(x), \qquad \alpha > 0, \tag{3}$$

$$\int_{|x|=1} k(x)d\sigma = 0. \tag{4}$$

Now if $k(x)$ is any function satisfying (3) and (4), together with a certain mild smoothness or boundedness condition [**1**; **2**], then the mapping

$$\to \int_{R^n} k(x-y)f(y)dy \equiv \lim_{\epsilon\to 0+} \int_{|x-y|>\epsilon} k(x-y)f(y)dy \tag{5}$$

is a continuous transformation of L^p into L^p, for $1<p<\infty$. In (2) the integral is also interpreted in the principal value sense of (5). The integral in (5) converges in L^p and pointwise almost everywhere. In particular, the derivatives $u_{x_ix_j}$ in (2) satisfy

$$\|u_{x_ix_j}\|_{L^p} \leqq A_p\|f\|_{L^p}.$$

3. **The equation** $u_t-\Delta u=f$. For $0<t<\infty$, $x\in R^n$, let $\Gamma(x, t)$ be the fundamental solution of the heat equation $(u_t-\Delta u=0)$,

$$\Gamma(x,t) = (4\pi t)^{-n/2}\exp\left(-\frac{|x|^2}{4t}\right).$$

Let

$$u(x,t) = \int_0^t\int_{R^n} \Gamma(x-y, t-s)f(y,s)dyds, \tag{1'}$$

where $f\in L^p(R^n\times(0,\infty))$. The analogue of (2) is

$$\begin{aligned} u_{x_ix_j} &= \int_0^t\int_{R^n} k_{ij}(x-y,t-s)f(y,s)dyds, \\ u_t &= f(x,t) + \int_0^t\int_{R^n} k'(x-y,t-s)f(y,s)dyds. \end{aligned} \tag{2'}$$

Here

$$k_{ij}(x,t) = \Gamma_{x_ix_j} = (4\pi)^{-n/2}t^{-n/2-1}\left(-\frac{1}{2}\delta_{ij} + \frac{x_ix_j}{4t}\right)\exp\left(-\frac{|x|^2}{4t}\right),$$

$$k'(x,t) = \Gamma_t = (4\pi)^{-n/2}t^{-n/2-1}\left(-\frac{1}{2}n + \frac{|x|^2}{4t}\right)\exp\left(-\frac{|x|^2}{4t}\right).$$

In particular, $u_t-\Delta u=f$.

Now let us consider kernels $k(x, t)$ satisfying

$$k(x, t) = 0, \qquad t < 0,$$

$$(3') \qquad k(\alpha x, \alpha^2 t) = \alpha^{-n-2} k(x, t), \qquad \alpha > 0,$$

$$(4') \qquad \int_{R^n} k(x, 1)dx = 0,$$

together with certain mild smoothness and boundedness conditions. For instance, it is sufficient to require that $|k(x, 1)| + |k_{x_i}(x, 1)| \leqq ae^{-b|x|}$ for some $a > 0$, $b > 0$. All these properties are satisfied by the kernels $k_{ij}(x, t)$ and $k'(x, t)$. Note the analogy between conditions (3′), (4′) and (3), (4), respectively.

If $k(x, t)$ is any such function, then we shall consider the mapping

$$(5') \qquad \begin{aligned} f \to \int_0^t \int_{R^n} k(x - y, t - s) f(y, s) dyds \\ \equiv \lim_{\epsilon \to 0+} \int_0^{t-\epsilon} \int_{R^n} k(x - y, t - s) f(y, s) dyds, \end{aligned}$$

the integral converging in L^p. The integrals in (2′) are also interpreted in this sense. Moreover the mapping (5′) defines a continuous transformation of L^p into L^p, $1 < p < \infty$. Thus, for example, the derivatives $u_{x_i x_j}$ and u_t in (2′) satisfy

$$\|u_{x_i x_j}\|_{L^p} + \|u_t\|_{L^p} \leqq A_p \|f\|_{L^p}.$$

The precise statements and proofs of these and allied results will appear elsewhere. The proofs are similar to some of the arguments in [1].

References

1. A. P. Calderón and A. Zygmund, *On the existence of certain singular integrals*, Acta Math. **88** (1952), 85–139.

2. ———, *On singular integrals*, Amer. J. Math. **78** (1956), 289–309.

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