COHOMOLOGY OF HOMOGENEOUS SPACES[1,2]

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Various authors have studied the following problem: "Let K be a field or the integers. If G is a compact connected Lie group and U is a closed connected subgroup how can the cohomology of the homogeneous space G/U, $H^*(G/U; K)$, be computed from $H^*(G; K)$, $H^*(U; K)$ and some algebraic topological invariant of the way U is imbedded in G?"

The most comprehensive results to date on this question have been obtained by H. Cartan [**3**] and A. Borel [**1**]. H. Cartan [**3**] solved the problem for the special case when the coefficient ring is the real numbers. A. Borel [**1**] essentially solved the problem for the special case when U is a subgroup of maximal rank and both $H^*(G; K)$ and $H^*(U; K)$ are exterior algebras on generators of odd degree. Indeed, Borel's work in [**1**], together with a result of R. Bott [**2**], gives a complete solution for this case.

For the invariant of the imbedding of U in G both Cartan and Borel take the cohomology map $\rho^*: H^*(B_G; K) \to H^*(B_U; K)$ induced by the map $\rho: B_U \to B_G$ of classifying spaces arising from the inclusion $U \subset G$. If $H^*(G; K)$ and $H^*(U; K)$ are both exterior algebras on generators of odd degree the results of [**1**] give a method for computing ρ^* from group-theoretic information on how U is imbedded in G.

Using unpublished results of S. Eilenberg and J. C. Moore the following generalization of the Cartan-Borel results is obtained:

THEOREM. *Let K be a field or the integers. Assume that $H^*(G; K)$ and $H^*(U; K)$ are exterior algebras on generators of odd degree. Consider $H^*(B_U; K)$ to be an $H^*(B_G; K)$ module by means of the map $\rho^*: H^*(B_G; K) \to H^*(B_U; K)$. Then the algebra structures in $H^*(B_G; K)$ and $H^*(B_U; K)$ induce an algebra structure in*

$$\mathrm{Tor}_{H^*(B_G;K)}(K, H^*(B_U; K))$$

such that for a suitable filtration of the algebra $H^*(G/U; K)$

$$\mathrm{Tor}_{H^*(B_G;K)}(K, H^*(B_U; K)) = E_0 H^*(G/U; K).$$

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[2] The work announced here is contained in the author's doctoral thesis, submitted to Princeton University. The author thanks his advisers J. C. Moore and N. E. Steenrod for the guidance and encouragement they gave him.

REMARKS. 1. The proof is independent of the Cartan-Borel results and new proofs of their results are obtained.

2. If the coefficient ring is the integers Z then

$$\operatorname{Tor}_{H^*(B_G;Z)}(Z,\ H^*(B_U;\ Z))$$

and $H^*(G/U;\ Z)$ are isomorphic as abelian groups.

3. If the coefficient ring is a field then the algebra structure of $\operatorname{Tor}_{H^*(B_G;K)}(K,\ H^*(B_U;\ K))$ can be closely analyzed and sufficient conditions for $\operatorname{Tor}_{H^*(B_G;K)}(K,\ H^*(B_U;\ K))$ and $H^*(G/U;\ K)$ to be isomorphic as algebras can be derived.

4. The hypotheses of the theorem are very frequently satisfied. If G is a Lie group with no p-torsion then $H^*(G;\ K)$ is an exterior algebra on generators of odd degree whenever K is a field of characteristic p. For example, $H^*(U(n);K)$, $H^*(SU(n);K)$ and $H^*(\mathrm{Sp}(n);K)$ are exterior algebras on generators of odd degree for any coefficient ring K. $H^*(SO(n);\ K)$ and $H^*(\mathrm{Spin}(n);\ K)$ are exterior algebras on generators of odd degree whenever K is a field whose characteristic is not 2. There are many examples where G and U are free of p-torsion but G/U has p-torsion.

5. A corollary of the theorem is:

COROLLARY. *Let σ denote the characteristic map of the principal U bundle $U\to G\to G/U$. Consider the sequence*

$$H^*(B_G;\ K) \xrightarrow{\rho^*} H^*(B_U;\ K) \xrightarrow{\sigma^*} H^*(G/U;\ K).$$

Whenever the hypotheses of the above theorem are satisfied the kernel of σ^* is the ideal of $H^*(B_U;\ K)$ generated by the elements of positive degree in Image ρ^*.

OUTLINE OF PROOF. Let $F\to^i E\to^\pi B$ be a fibration in the sense of Serre. Assume that F, E, and B are connected and that B is simply connected. Assume also that for each integer q, $H^q(E;K)$ and $H^q(B;K)$ are finitely generated K-modules. Consider $H^*(E;\ K)$ to be an $H^*(B;\ K)$ module by means of the map $\pi^*\colon H^*(B;\ K)\to H^*(E;\ K)$. In this situation S. Eilenberg and J. C. Moore have constructed (unpublished) a spectral sequence converging to $H^*(F;\ K)$ whose E_2 term is $\operatorname{Tor}_{H^*(B;K)}(K,\ H^*(E;\ K))$. The method of proof is to apply this Eilenberg-Moore spectral sequence to the fibration

$$G/U \xrightarrow{\sigma} B_U \xrightarrow{\rho} B_G$$

and show that whenever the hypotheses of the theorem are satisfied this spectral sequence has $E_2=E_\infty$.

The case where the coefficient ring is the integers follows from the field case by a universal coefficient argument. Thus it suffices to consider the case when the coefficient ring K is a field, so from now on K is a field.

The special case when U is a subgroup of maximal rank is proved by applying some algebraic results on E-sequences [4]. Using the maximal rank result it is then shown that it suffices to prove the theorem for the case when the subgroup is a torus.

The torus case is proved by induction on the dimension of the torus. If the torus is a zero dimensional torus, i.e. if the torus is just the identity element of the group G, then the fibration to be studied is just $G \to E_G \to B_G$, the universal G-fibration. An explicit calculation shows that $\mathrm{Tor}_{H^*(B_G;K)}(K, K)$ and $H^*(G; K)$ are isomorphic as algebras.

Now let T_{l-1} and T_l be respectively an $l-1$ and an l dimensional torus of G with $T_{l-1} \subset T_l$. A commutative diagram

$$\begin{array}{ccccc} B_{T_l/T_{l-1}} & \to & B_{T_l/T_{l-1}} & \to & \cdot \\ \uparrow & & \uparrow & & \uparrow \\ G/T_l & \xrightarrow{\sigma} & B_{T_l} & \xrightarrow{\rho} & B_G \\ \uparrow & & \uparrow & & \uparrow \\ G/T_{l-1} & \xrightarrow{\sigma'} & B_{T_{l-1}} & \xrightarrow{\rho'} & B_G \end{array}$$

is constructed in which each row and each column is a fibration. It is shown that if the Eilenberg-Moore spectral sequence of the bottom row has $E_2 = E_\infty$, then so does the Eilenberg-Moore spectral sequence of the middle row. This completes the inductive step.

Full details will be published elsewhere.

References

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