ON LINKED BINARY REPRESENTATIONS OF PAIRS OF INTEGERS: SOME THEOREMS OF THE ROMANOV TYPE[1]

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1. **Introduction.** Let us denote by N the sequence $\{1, 2, 3, \cdots\}$, by p a prime, by (a, b) the greatest common divisor of a and b, by $[a, b]$ the least common multiple of a and b, by $\{*: \cdots\}$ resp. $A\{*: \cdots\}$ the set resp. number of $*$ with the properties $\cdots$, by μ the Moebius function, by C an absolute positive constant and by $C(*)$ a positive constant depending on $*$ only.

Suppose $N_j \subset N$ $(j=1, 2, 3, 4)$ and denote by $y_1 \sim y_2$ an arbitrary relation (= linking) with $y_{1,2} \in N$. For instance, $[y_1 \sim y_2]: = [(y_1, y_2)=1]$ resp. $[y_1 \sim y_2]:=[y_1=y_2]$ can be considered a weak resp. strong linking. By a linked binary representation of a pair m, n with $m \in N$ and $n \in N$ we mean a solution x_1, x_2, x_3, x_4 of the Diophantine system $x_1+x_2=m \wedge x_3+x_4=n \wedge x_j \in N_j$ $(j=1, 2, 3, 4) \wedge x_2 \sim x_4$. Various generalizations are obvious (more summands, triples, etc.). We do not intend to give a detailed and general study of the questions arising in this context. We rather prefer to investigate two special problems of this type with $\sim$ being $=$; they are inspired by the following two well-known results of Romanov:

$$E_a: = \{m: m = p + v^a \wedge p \text{ prime} \wedge v \in N\} \qquad (1 < a \in N)$$

and

$$F_a: = \{m: m = p + a^v \wedge p \text{ prime} \wedge v \in N\} \qquad (a \in N)$$

have positive asymptotic density [1, pp. 63–70].

2. **On Romanov's first theorem.** Generalizing the result for E_a, we show that the set $\{m, n: m=p_1+v^a \wedge n=p_2+v^a \wedge p_{1,2} \text{ prime} \wedge v \in N\}$, considered as a set of lattice points in the plane, has positive asymptotic density in the plane:

THEOREM 1. *For $1<a\in N$ there exist constants $C_1(a)$ and $C_2(a)$ such that $x>C_1(a)$ implies*

$$A_1(x, a): = A\{m, n: m < x \wedge n < x \wedge m = p_1 + v^a \wedge n = p_2 + v^a \wedge p_{1,2} \text{ prime} \wedge v \in N\} > C_2(a)\, x^2.$$

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PROOF. Let $f_1(m, n; a) := A\{p_1, p_2, v: p_1 + v^a = m \wedge p_2 + v^a = n\}$; since $A_1(x, a) = A\{m, n: m < x \wedge n < x \wedge f_1(m, n; a) > 0\}$, the Schwarz inequality yields

$$\left(\sum_{m<x}\sum_{n<x} f_1(m, n; a)\right)^2 \leqq A_1(x, a) \sum_{m<x}\sum_{n<x} f_1^2(m, n; a). \tag{1}$$

On the one hand, we find

$$\begin{aligned} \sum_{m<x}\sum_{n<x} f_1(m, n; a) &= A\{p_1, p_2, v: p_1 + v^a < x \wedge p_2 + v^a < x\} \\ &\geqq A\left\{p_1: p_1 < \frac{x}{2}\right\} A\left\{p_2: p_2 < \frac{x}{2}\right\} A\left\{v: v^a < \frac{x}{2}\right\} \\ &> C_4\left(\frac{x}{\log x}\right)^2\left(\frac{x}{2}\right)^{1/a} \qquad (x > C_3(a)). \end{aligned} \tag{2}$$

On the other hand, we find

$$\begin{aligned} S_1(x, a) :&= \sum_{m<x}\sum_{n<x} f_1^2(m, n; a) \\ &= A\{p_1, p_2, p_3, p_4, v_1, v_2: p_1 + v_1^a = p_2 + v_2^a < x \wedge p_3 + v_1^a \\ &\qquad\qquad = p_4 + v_2^a < x\} \\ &\leqq \sum_{v_1 < x^{1/a}} \sum_{v_2 < x^{1/a}} A\{p_1, p_2, p_3, p_4: p_1 - p_2 = p_3 - p_4 \\ &\qquad\qquad = v_2^a - v_1^a \wedge p_{1,2,3,4} < x\}. \end{aligned}$$

In case of $v_1 = v_2$ resp. $v_1 \neq v_2$ we use

$$A\{p: p < x\} < C_5 \frac{x}{\log x} \qquad (x > 2)$$

resp. Brun's sieve method [2, 2. Satz 4.2] and obtain

$$S_1(x, a) < C_7\left(\frac{x}{\log x}\right)^2 x^{1/a} + 2 \sum_{v_2 < v_1 < x^{1/a}} \left(C_8 \frac{x}{\log^2 x} g(v_1^a - v_2^a)\right)^2 \qquad (x > C_6)$$

where

$$g(b) := \prod_{p|b}\left(1 + \frac{1}{p}\right) = \sum_{d|b;\, \mu(d) \neq 0} \frac{1}{d}\,.$$

It follows

$$S_1(x, a) < C_7\left(\frac{x}{\log x}\right)^2 x^{1/a} + C_9 \frac{x^2}{\log^4 x} \sum_{u<x} F(u; x, a) g^2(u) \qquad (x > C_6)$$

where

$$F(u; x, a) := A\{v_1, v_2 : v_2 < v_1 < x^{1/a} \wedge v_1^a - v_2^a = u\}.$$

Writing $g(u)$ as a sum and changing the order of summation gives

$$\sum_{u<x} F(u; x, a) g^2(u) = \sum_{\substack{d_1<x \\ \mu(d_1)\neq 0}} \sum_{\substack{d_2<x \\ \mu(d_2)\neq 0}} \frac{1}{d_1 d_2} B([d_1, d_2]; x, a)$$

where

$$B(k; x, a) := \sum_{u<x;\, u\equiv 0 \bmod k} F(u; x, a) < 2x^{2/a} k^{-1/a} a^{w(k)} \qquad (\mu(k) \neq 0)$$

[1, p. 66] with

$$w(k) := A\{p : p \mid k\} < C_{10} \frac{\log k}{\log \log k}.$$

Since $\mu(d_1)\neq 0 \wedge \mu(d_2) \neq 0$ imply $\mu([d_1, d_2]) \neq 0$, we obtain

$$S_1(x, a) < C_7 \frac{x^{2+1/a}}{\log^2 x} + \frac{x^{2+2/a}}{\log^4 x} C_{11}(a) \sum_{\substack{d_1<x \\ \mu(d_1)\neq 0}} \sum_{\substack{d_2<x \\ \mu(d_2)\neq 0}} (d_1 d_2)^{-1} [d_1, d_2]^{-1/2a} \qquad (x > C_6).$$

Using $[d_1, d_2]^2 \geqq d_1 d_2$, we find

$$S_1(x, a) < C_{12}(a) \frac{x^{2+2/a}}{\log^4 x} \qquad (x > C_6). \tag{3}$$

(1), (2), and (3) give the desired result.

It is not difficult to determine a dependence of $C_{1,2}(a)$ on a explicitly. Since $A_1(x, a) \leqq A\{m, n : m < x \wedge n < x\}$, Theorem 1 is best possible with respect to the order of magnitude in x. Theorem 1 is also correct for $a = 1$ but of no interest.

3. **On Romanov's second theorem.** In a similar way we generalize the result for F_a:

THEOREM 2. *For $1 < a \in N$ there exist constants $C_{13}(a)$ and $C_{14}(a)$ such that $x > C_{13}(a)$ implies*

$$A_2(x, a) := A\{m, n : m < x \wedge n < x \wedge m = p_1 + a^v \wedge n = p_2 + a^v \wedge p_{1,2} \text{ prime} \wedge v \in N\} > C_{14}(a) \frac{x^2}{\log x}.$$

PROOF. Let $f_2(m, n; a) := A\{p_1, p_2, v : p_1 + a^v = m \wedge p_2 + a^v = n\}$. As in the preceding proof, we find

$$\sum_{m<x} \sum_{n<x} f_2(m, n; a) > C_{16}\left(\frac{x}{\log x}\right)^2 \frac{\log x/2}{\log a} \qquad (x > C_{15}(a)) \tag{4}$$

and

$$\begin{aligned} S_2(x, a) := & \sum_{m<x} \sum_{n<x} f_2^2(m, n; a) \\ & < C_{18}\left(\frac{x}{\log x}\right)^2 \frac{\log x}{\log a} + 2 \sum_{v_2<v_1<\log x/\log a}\sum \left(C_8 \frac{x}{\log^2 x} g(a^{v_1} - a^{v_2})\right)^2 \\ & \qquad\qquad (x > C_{17}). \end{aligned}$$

For $v_1 > v_2$ we have

$$g(a^{v_1} - a^{v_2}) = g(a)g(a^{v_1-v_2} - 1);$$

with $h: = v_1 - v_2$ we get

$$\begin{aligned} S_2(x, a) < & C_{19}(a) \frac{x^2}{\log x} \\ & + 2\left(C_8 \frac{x}{\log^2 x}\right)^2 \frac{\log x}{\log a} \sum_{h<\log x/\log a} g^2(a^h - 1) \qquad (x > C_{17}). \end{aligned}$$

For $(a, d) = 1$, let $e(a, d)$ denote the exponent of a mod d (i.e., the certainly existing smallest $t \in N$ with $a^t \equiv 1 \bmod d$); then $d \mid (a^h - 1)$ implies $(a, d) = 1 \wedge e(a, d) \mid h$. Therefore,

$$\begin{aligned} \sum_{h<\log x/\log a} g^2(a^h - 1) = & \sum_{h<\log x/\log a} \sum_{\substack{d_1 \mid (a^h-1) \\ \mu(d_1)\neq 0}} \frac{1}{d_1} \sum_{\substack{d_2 \mid (a^h-1) \\ \mu(d_2)\neq 0}} \frac{1}{d_2} \\ \leqq & \sum_{\substack{d_1<x \\ \mu(d_1)\neq 0 \\ (d_1,a)=1}} \sum_{\substack{d_2<x \\ \mu(d_2)\neq 0 \\ (d_2,a)=1}} \frac{1}{d_1 d_2} \sum_{\substack{h<\log x/\log a \\ h\equiv 0 \bmod e(a,d_1) \\ h\equiv 0 \bmod e(a,d_2)}} 1 \\ \leqq & \frac{\log x}{\log a} \sum_{\substack{d_1<x \\ \mu(d_1)\neq 0 \\ (d_1,a)=1}} \sum_{\substack{d_2<x \\ \mu(d_2)\neq 0 \\ (d_2,a)=1}} \frac{1}{d_1 d_2 [e(a, d_1), e(a, d_2)]} \\ \leqq & \frac{\log x}{\log a} \left(\sum_{\substack{d<x \\ \mu(d)\neq 0 \\ (d,a)=1}} d^{-1}(e(a, d))^{-1/2} \right)^2 < C_{20}(a) \log x, \end{aligned}$$

since $[a, b]^2 \geqq ab$ and since, for an arbitrary positive increasing function f,

$$\sum_{d=1} \frac{1}{d f(d)} < \infty$$

implies

$$\sum_{(d,a)=1;\mu(d)\neq 0} \frac{1}{d f(e(a,d))} < C_{21}(a,f)$$

[3, Satz 3]. Hence, we have

$$S_2(x,a) < C_{22}(a)\frac{x^2}{\log x} \qquad (x > C_{17}). \tag{5}$$

(4), (5), and (1) with index 2 instead of 1 give the desired result.

It is not difficult to give an explicit dependence of $C_{13}(a)$ and $C_{14}(a)$ on a. Again, since

$$\begin{aligned} A_2(x,a) &\leqq A\{p_1, p_2, v: p_{1,2} < x \wedge a^v < x\} \\ &< \left(C_5 \frac{x}{\log x}\right)^2 \frac{\log x}{\log a} \qquad (x > 2), \end{aligned}$$

Theorem 2 is best possible in x.

4. Generalization to algebraic number fields K. For convenience, let K be a totally real algebraic number field. Denote by n the degree of K, by $J(K)$ the ring of all integers of K, by small Greek letters elements of $J(K)$, by $\xi^{(1)}, \cdots, \xi^{(n)}$ the conjugates of ξ, and by $\xi \prec x$ the system $|\xi^{(j)}| < x$ $(j=1, \cdots, n)$. π is called a prime if π generates a prime ideal of $J(K)$. Combining the method used above with ideas of [4], we arrive at direct generalizations of Theorem 1 and Theorem 2:

THEOREM 1'. *For $1 < a \in N$ there exist constants $C_{23}(K, a)$ and $C_{24}(K, a)$ such that $x > C_{23}(K, a)$ implies*

$$A\{\sigma, \tau: \sigma = \pi_1 + \nu^a \wedge \tau = \pi_2 + \nu^a \wedge \pi_{1,2} \text{ prime} \wedge \pi_{1,2} \prec x \wedge \nu \prec x^{1/a}\} > C_{24}(K,a)x^{2n}.$$

THEOREM 2'. *For $0 \neq \alpha \in J(K)$ and not a root of unity there exist constants $C_{25}(K, \alpha)$ and $C_{26}(K, \alpha)$ such that $x > C_{25}(K, \alpha)$ implies*

$$A\{\sigma, \tau: \sigma = \pi_1 + \alpha^v \wedge \tau = \pi_2 + \alpha^v \wedge \pi_{1,2} \text{ prime} \wedge \pi_{1,2} \prec x \wedge v \in N \wedge \alpha^v \prec x\} > C_{26}(K,\alpha)\frac{x^{2n}}{\log x}.$$

Again, the estimates are best possible in x.

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THE COHOMOLOGY OF CERTAIN ORBIT SPACES[1]

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Let (G, X) be a topological transformation group—or action—in which G is finite and X is locally compact. An important part of the cohomology of the orbit space X/G lies, so to speak, in the free part $\mathfrak{f}$ of the action (i.e. the union of orbits of cardinality $[G:1]$). The cohomology of $\mathfrak{f}/G$ can be regarded as an $H(G)$-module. We shall exhibit a complete set of generators and relations for this module assuming G to be the direct product of cyclic groups of prime order p and X to be a generalized sphere over Z_p (see [4, p. 404]). H will always denote cohomology with values in Z_p. A useful device consists in relating the generators of $H(G)$ to those of G.

Dimension functions. From now on let $G=Z_p\times \cdots \times Z_p$, r factors, and let $\mathfrak{g}_i$ be the collection of subgroups of order p^i; $\mathfrak{g}_0$ consists of the identity only. Let $g, h, \cdots$ always denote subgroups of G and $g_i, h_i, \cdots$ elements of $\mathfrak{g}_i$. In particular $g_0=\{1\}$ and $g_r=G$.

By a *dimension function* of the pair (G, p) we shall mean an integer-valued function $n(g)$ of constant parity with values $\geqq -1$ and such that for each g different from G

$$n(g) = n(G) + \sum_h (n(h) - n(G)) \tag{1}$$

summed over those h's which lie in $\mathfrak{g}_{r-1}$ and contain g; when $p=2$, constant parity is not required.

For a given dimension function $n(g)$ let Ω be the totality of se-

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