References

1. E. Landau, *Ueber einige neuere Fortschritte der additiven Zahlentheorie*, Cambridge Univ. Press, Cambridge, 1937.
2. Karl Prachar, *Primzahlverteilung*, Springer, Berlin, 1957.
3. G. J. Rieger, *Verallgemeinerung eines Sätzes von Romanov und anderes*, Math. Nachr. 20 (1959), 107–122.
4. ———, *Verallgemeinerung zweier Sätze von Romanov aus der additiven Zahlentheorie*, Math. Ann. 144 (1961), 49–54.

Purdue University and
University of Munich

THE COHOMOLOGY OF CERTAIN ORBIT SPACES[1]

BY P. A. SMITH

Communicated by Deane Montgomery, March 11, 1963

Let (G, X) be a topological transformation group—or action—in which G is finite and X is locally compact. An important part of the cohomology of the orbit space X/G lies, so to speak, in the free part $\mathfrak{f}$ of the action (i.e. the union of orbits of cardinality $[G:1]$). The cohomology of $\mathfrak{f}/G$ can be regarded as an $H(G)$-module. We shall exhibit a complete set of generators and relations for this module assuming G to be the direct product of cyclic groups of prime order p and X to be a generalized sphere over Z_p (see [4, p. 404]). H will always denote cohomology with values in Z_p. A useful device consists in relating the generators of $H(G)$ to those of G.

Dimension functions. From now on let $G = Z_p \times \cdots \times Z_p$, r factors, and let $\mathfrak{g}_i$ be the collection of subgroups of order p^i; $\mathfrak{g}_0$ consists of the identity only. Let $g, h, \cdots$ always denote subgroups of G and $g_i, h_i, \cdots$ elements of $\mathfrak{g}_i$. In particular $g_0 = \{1\}$ and $g_r = G$.

By a *dimension function* of the pair (G, p) we shall mean an integer-valued function $n(g)$ of constant parity with values $\geqq -1$ and such that for each g different from G

$$n(g) = n(G) + \sum_h (n(h) - n(G)) \tag{1}$$

summed over those h's which lie in $\mathfrak{g}_{r-1}$ and contain g; when $p = 2$, constant parity is not required.

For a given dimension function $n(g)$ let Ω be the totality of se-

[1] This work has been supported by the Office of Naval Research.

quences $\omega=(g_1, \cdots, g_r)$ such that $g_i \subset g_{i+1}$, $n(g_i)>n(g_{i+1})$, $i=1, \cdots, r-1$. Call $n(g)$ *effective* if Ω is nonempty.

In any action (G, X) denote by $F(g)$ the fixed-point set of g. If X is a generalized N-sphere over Z_p then $F(g)$ is a generalized k-sphere over Z_p with $k=-1$ if $F(g)$ is empty. It is known [1, Chapter XII] that the function $n(g)=k$ is a dimension function with $n(g_0)=N$. We shall call this $n(g)$ the dimension function of (G, X); it is effective if and only if (G, X) is effective (i.e. if the free part is nonempty). The relations (1) were obtained by Borel [1, Chapter XIII].

Generators and relations. Let $n(g)$ be an effective dimension function and let $\boldsymbol{F}$ be the free $H(G)$-module which has the elements of Ω as free generators. The elements of $\boldsymbol{F}$ are of the form $\sum_\omega \lambda_\omega \omega$ where $\lambda_\omega \in H(G)$. We grade $\boldsymbol{F}$ by assigning the degree $n(G)+r$ to each ω and taking $\deg \lambda_\omega\omega = \deg \lambda_\omega + \deg \omega$. We shall define subsets R_1, R_2 of $\boldsymbol{F}$. R_1 consists of all elements

$$\sum_{h_i} (g_1, \cdots, h_i, \cdots, g_r) \qquad (i = 1, \cdots, r-1) \tag{2}$$

where the g's are such that the set g_i' of h_i's for which $(g_1, \cdots, h_i, \cdots, g) \in \Omega$ is not empty, and the summation is over g_i'. To define R_2, choose a fixed base $t=(t_1, \cdots, t_r)$ for G and for each $\omega=(g_1, \cdots, g_r) \in \Omega$ choose a base $s(\omega)$ *associated with* ω, namely a base $(s_1, \cdots, s_r)$ of G the first i elements of which generate g_i $(i=1, \cdots, r)$. In addition choose a base for $H(G)$, namely a set of elements u_i of degree 1 and v_i of degree 2 $(i=1, \cdots, r)$ such that $\{u_i, v_i\}$ generates the algebra $H(G)$ (when $p=2$, a base consists of r elements of degree 1 only). The s_i in $s(\omega)$ are uniquely expressible as $s_i=t_1^{p_{i1}} \cdots t_r^{p_{ir}}$. Let $P(\omega)$ be the matrix (p_{ij}). We shall define R_2 only for the case $p>2$: it consists of all elements

$$V_i^{a_i} U_{i+1} V_{i+1}^{b_{i+1}} \cdots U_r V_r^{b_r} \omega \qquad (i = 1, \cdots, r; \omega \in \Omega). \tag{3}$$

The U's and V's depend on ω and are given by

$$U_j = \sum q_{jk} u_k, \qquad V_j = \sum q_{jk} v_k$$

where $Q(\omega)=(q_{jk})$ is the transposed inverse of $P(\omega)$. The a's and b's are:

$$a_i = a_i(\omega) = \tfrac{1}{2}(n(g_{r-i-1}) - n(g_{r-i})), \quad b_i = a_i - 1 \quad (i = 1, \cdots, r).$$

Let $A_{n(g)}=\boldsymbol{F}/\boldsymbol{R}$ where $\boldsymbol{R}$ is generated additively by $R_1 \cup R_2$. It is easily shown that $A_{n(g)}$ depends only on $n(g)$.

THEOREM. *Let $n(g)$ be the dimension function of an effective action*

(G, X) where X is a generalized sphere over Z_p and let $\mathfrak{f}$ be the free part of the action. The $H(G)$-modules $A_{n(g)}$ and $H(\mathfrak{f}/G)$ are isomorphic. Every effective dimension function $n(g)$ is the dimension function of an effective orthogonal action (G, S) where $N=n(g_0)$.

Remark. The elements in $R_1 \cup R_2$ are not necessarily linearly independent and therefore $R_1 \cup R_2$ can generally be replaced by a proper subset. It can be shown for example that when $r=2$, the index i in (3) need only take the value 1. Thus when $r=2$, we may take for R_2 the elements $V_1^{a_1}\omega$, $\omega \in \Omega$.

An example. Let $r=2$, $p=3$. $\mathfrak{g}_1$ consists of four subgroups,—call them $g^1, \cdots, g^4$ omitting the subscript 1. Let $n(g_0)=9$, $n(g^1) = n(g^2)=n(g^3)=1$, $n(g^4)=3$, $n(g_2)=n(G)=-1$. This defines a dimension function $n(g)$ for G and $n(g)$ is effective: $\Omega=\{\omega_i\}$ where $\omega_i=(g^i, g_2)$, $i=1, \cdots, 4$. R_1 consists of the single element $\omega_1+\omega_2+\omega_3+\omega_4$. We take R_2 as in the Remark. Simple calculations give $R_2 = \{v_2\omega_1, (-v_1+v_2)\omega_2, (v_1+v_2)\omega_3, v_1^2\omega_4\}$. Evidently $A_{n(g)}=\boldsymbol{F}'/\boldsymbol{R}'$ where $\boldsymbol{F}'$ is generated by ω_1, ω_2, ω_3 and $\boldsymbol{R}'$ by

$$(4) \qquad v_2\omega_1, \quad (-v_1+v_2)\omega_2, \quad (v_1+v_2)\omega_3, \quad v_1^2(\omega_1+\omega_2+\omega_3).$$

It can be verified that $A_{n(g)}$ agrees with known formulas [**3**] for $H(\mathfrak{f}/G)$. It is known for example that $H^n(\mathfrak{f}/G)$ is cyclic when $n=n(g_0)$ and is trivial for larger n. To see how this works out in the present example where $n(g_0)=9$, one verifies first that all elements $\lambda\omega_i$ in $\boldsymbol{F}'$ of degree greater than 7 in which λ is a polynomial in v_1, v_2 alone, is in $\boldsymbol{R}'$. For example

$$-v_1^4\omega_1 = v_1^2 v_2 r_1 + (v_1^3 + v_1^2 v_2) r_2 + (v_1^2 v_2 - v_1^3) r_3 + (v_1^2 - v_2^2) r_4,$$

where $r_1, \cdots$ are the elements in (4). Now $u_1^2=u_2^2=0$ since the u's are of odd degree. It follows readily that all elements of $\boldsymbol{F}'$ of degree >9 are in $\boldsymbol{R}'$. As for the degree 9, it is easily shown that the element $e=u_1u_2v_1^3\omega_1$ is not in $\boldsymbol{R}'$. Moreover if e_1 is of degree 9 then $e_1=xe$ mod $\boldsymbol{R}'$ where $x \in Z_p$. For example

$$u_1u_2v_1^3\omega_2 = u_1u_2v_1^3\omega_1 + u_1u_2(v_1^2 r_1 + (v_2^2 + v_1v_2 - v_1^2) r_2 v_1^2 r_3 - (v_1+v_2) r_4).$$

The structure of $A_{n(g)}$ in lower degrees can be determined just as readily and compared with the results in [**3**].

***$H(G)$*-modules.** Let $C=(C^n)_{n\geq 0}$ be the graded Z_p-module in which each C^n is the group ring $Z_p(Z_p)$. Let $t=(t_1, \cdots, t_r)$ be a base for G and let

$$C_t = C_{(1)} \otimes \cdots \otimes C_{(r)}$$

where the $C_{(i)}$ are copies of C. Convert C_t to a G-module by the action

$$\gamma(c_1 \otimes \cdots \otimes c_r) = \gamma_1 c_1 \otimes \cdots \otimes \gamma_r c_r \qquad (\gamma \in G)$$

where γ_i is the t_i-component of γ. Define coboundaries in C_t by taking $d_i^n\colon C_{(i)}^n \to C_{(i)}^{n+1}$ to be multiplication by $\sigma(t_i) = 1 + t_i + \cdots + t_i^{p-1}$ when n is odd and by $1 - t_i$ when n is even. C_t is now a free acyclic G-complex of cochains. Let

$$u_i(t) = 1 \otimes \cdots \otimes \sigma(t_i) \otimes \cdots \otimes 1 \qquad (i = 1, \cdots, r)$$

where the 1's are of degree zero, $\sigma(t_i)$ of degree 1 and let $v_i(t)$ be the same except that $\sigma(t_i)$ is of degree 2. On introducing products in the usual way [**2**, p. 252] C_t becomes an algebra which induces an algebra $H(C_t)^G$ where $(C_t)^G$ consists of the invariant elements of C_t. The elements of $H(C_t)^G$ represented by $\{u_i(t), v_i(t)\}$ (by $\{u_i(t)\}$ only when $p=2$) form a base β_t. Let $s = (s_1, \cdots, s_r)$ be a base for G. Now let G act diagonally on $C_s \otimes C_t$. The equivariant map $C_t \to C_s \otimes C_t$ defined by $c_t \to \epsilon(s) \otimes c_t$ where $\epsilon(s) = 1 \otimes \cdots \otimes 1 \in C_s^0$, is known to induce an algebra isomorphism $H(C_t)^G \to H(C_s \otimes C_t)^G$ and there is also an isomorphism $H(C_s)^G \to H(C_s \otimes C_t)^G$. We introduce into $\bigcup_t H(C_t)$ an equivalence which is compatible with multiplication: if $a \in H(C_s)^G$, $b \in H(C_t)^G$ then $a \sim b$ if a and b have equal images in $H(C_s \otimes C_t)^G = H(C_t \otimes C_s)^G$. We take for $H(G)$ the algebra of equivalence classes. There is a canonical isomorphism $\phi_t\colon H(C_t)^G \to H(G)$ for every t. The images in $H(G)$ of the elements of β_t give a base $\beta_t = \{u_i(t), v_i(t)\}$ for $H(G)$.

PROPOSITION 1. *Let s, t be bases for G and let $s_i = t^{p_{i1}} \cdots t^{p_{ir}}$, $i = 1, \cdots, r$. Then $u_i(s) = \sum q_{ij} u_j(t)$, $v_i(s) = \sum q_{ij} v_j(t)$ where $Q = (q_{ij})$ is the transposed inverse of $P = (p_{ij})$. (When $p = 2$, the v's do not appear.)*

The orbit space of f. Let (G, X) be an action and let $\mathbf{C}'(X)$ be the Alexander-Spanier cochains of X with values in Z_p modulo those with empty supports. Let $\mathbf{C}(X)$ be the compactly supported elements of $\mathbf{C}'(X)$. Let f be the free part of the action. $\mathbf{C}(\mathrm{f})$ is a free G-module and $H(\mathrm{f}/G)$ can be identified with $H(\mathbf{C}(\mathrm{f}))^G$. The map $\psi_t\colon \mathbf{C}(\mathrm{f}) \to C_t \otimes \mathbf{C}(\mathrm{f})$ defined by $\mathbf{c} \to \epsilon(t) \otimes \mathbf{c}$ induces an isomorphism $H(\mathrm{f}/G) \to H(C_t \otimes \mathbf{C}(\mathrm{f}))^G$. Thus an element x in $H(\mathrm{f}/G)$ can be regarded as an equivariant cohomology class of $C_t \otimes \mathbf{C}(\mathrm{f})$. It can be verified that the map $C_t \otimes (C_t \otimes \mathbf{C}(\mathrm{f})) \to C_t \otimes \mathbf{C}(\mathrm{f})$ defined by $c \otimes (c' \otimes x) \to cc' \otimes x$ $(c, c' \in C_t,\ x \in \mathbf{C}(\mathrm{f}))$ induces an action by $H(C_t)^G$ on

$H(C_t \otimes \mathbf{C}(\mathrm{f}))^G$, hence on $H(\mathrm{f}/G)$ such that $H(\mathrm{f}/G)$ is an $H(C_t)^G$-module. Through the isomorphism ψ_t, $H(\mathrm{f}/G)$ becomes an $H(G)$-module; the action by $H(G)$ on $H(\mathrm{f}/G)$ is independent of t.

Derivation of R_2. For each subgroup g of G there is an induced action $(G/g, F(g))$. We denote its free part by $\mathrm{f}(g)$ agreeing that $\mathrm{f}(g_r) = F(g_r)$. Now assume that (G, X) is effective and let $\omega = (g_1, \cdots, g_r)$ be an element of Ω. Let $s(\omega) = (s_1, \cdots, s_r)$ be associated with ω and let h_j be the subgroup generated by $s^j = (s_{r-j+1}, \cdots, s_r)$. The induced action $(h_j, F(g_{r-j}))$ can be identified with $(G/g_{r-j}, F(g_{r-j}))$ and hence $\mathrm{f}(g_{r-j})$ is the free part of $(h_j, F(g_{r-j}))$ and $H(\mathrm{f}(g_{r-j})/h_j)$ is an $H(h_j)$-module.

PROPOSITION 2. *There exist maps of degree* 1

$$H(\mathrm{f}(g_r)) \xrightarrow{\alpha_r} H(\mathrm{f}(g_{r-1})/h_1) \xrightarrow{\alpha_{r-1}} H(\mathrm{f}(g_{r-2})/h_2) \rightarrow \cdots \rightarrow H(\mathrm{f}/G)$$

such that for $x \in H(\mathrm{f}(g_{r-j})/h_j)$,

$$\alpha_{r-j} u_i(s^j)x = u_j(s^{j+1})\alpha_{r-j}x, \qquad j = 0, \cdots, r-1, i = 1, \cdots j. \tag{5}$$

If $p > 2$, *the same relation holds with* u *replaced by* v.

The α's are essentially connecting homomorphisms in the cohomology sequences for certain pairs.

PROPOSITION 3. *Let* t *be a fixed base for* G *and* (G, X) *be an action. Let* $n(g) \geqq -1$ *be an integer-valued function, of constant parity if* $p > 2$, *and such that* $F(g) = \varnothing$ *when* $n(g) = -1$ *and* $H^n(\mathrm{f}(g)/h) = 0$ $(h = G/g)$ *whenever* $n > n(g)$. *Let* $g_1 \subset g_2 \subset \cdots \subset g_r$ *be subgroups of* G *and* η *an element of* $H^{n(g_r)}(\mathrm{f}(g_r))$ *and let*

$$w(g_1, \cdots, g_r) = \alpha_1 \cdots \alpha_r \eta. \tag{6}$$

The expressions (3) *are annulled when* ω *is replaced by the corresponding* w *and* u_i *by* $u_i(t)$, v_i *by* $v_i(t)$.

PROOF FOR THE CASE $p > 2$. It will be seen that

$$y = v_i(s^i)^{a_i}\alpha_i u_{i+1}(s^{i+1})v_{i+1}(s^{i+1})^{b_{i+1}}\alpha_{i+1} \cdots \alpha_r u_r(s^{r-1})v_{r-1}(s_{r-1})^{b_{r-1}}\alpha_r\eta$$

lies in $H(\mathrm{f}(g_{r-1})/h_{r-i+1})$ and is of degree

$$2(a_i + b_{i+1} \cdots + b_r) + (r - i) + (r - i + 1) = n(g_{i-1}) + 1$$

hence is zero. Hence $\alpha_1 \cdots \alpha_{i-1} y = 0$. On transferring the α's to the right by (5) we obtain

$$v_i(s)^{a_i}u_{i+1}(s) \cdots u_r(s)v_r(s)^{b_r}w = 0.$$

From Proposition 1 we have $v_i(s)=V_i$, $u_{i+1}(s)=U_{i+1}$, etc. which completes the proof.

Now let X be a generalized sphere over Z_p and let $n(g)$ be the dimension function of (G, X) assumed to be effective. Let η be a nonzero element of $H^{n(g_r)}(\mathfrak{f}(G))$ and let

$$W = \{w(g_1 \cdots g_r), (g_1, \cdots, g_r) \in \Omega\}$$

where w is given by (6). (In case $F(g)=\emptyset$, replace $\alpha_r\eta$ by any nonzero element of $H^0(g_{r-1})$.) It can be shown that W generates the $H(G)$-module $H(F/G)$ and that its elements annul the expressions (2) when substituted for the corresponding ω's (the proof of this last does not involve $H(G)$). From Proposition 3 the expressions (3) are also annulled. It follows that there is a natural homomorphism $\mu: A_{n(g)} \rightarrow H(\mathfrak{f}/G)$ which is surjective. An argument based on [**3**] shows that μ is also injective.

REFERENCES

1. A. Borel, *Seminar on transformation groups*, Annals of Mathematics Studies No. 46, Princeton Univ. Press, Princeton, N. J., 1960.

2. H. Cartan and S. Eilenberg, *Homological algebra*, Princeton Univ. Press, Princeton, N. J., 1956.

3. P. A. Smith, *Orbit spaces of finite abelian transformation groups*, Proc. Nat. Acad. Sci. U.S.A. **47** (1961), 1662–1667.

4. ———, *New results and old problems in finite transformation groups*, Bull. Amer. Math. Soc. **66** (1960), 401–415.

COLUMBIA UNIVERSITY