MAXIMAL FUCHSIAN GROUPS

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1. DEFINITIONS. Let D be the unit disk $\{z \mid |z| < 1\}$ and let $\mathfrak{L}$ be the group of conformal homeomorphisms of D. A Fuchsian group is a discrete subgroup of $\mathfrak{L}$. We shall be concerned here with the finitely generated Fuchsian groups. It is known that these have the following presentations.

Generators: $a_1, b_1, \cdots, a_g, b_g, e_1, \cdots, e_k, h_1, \cdots, h_m, p_1, \cdots, p_r$.

Defining relations: $e_1^{\nu_1} = e_2^{\nu_2} = \cdots = e_k^{\nu_k} = 1$,

$$\left(\prod_{i=1}^{g} a_i b_i a_i^{-1} b_i^{-1}\right) e_1 \cdots e_k h_1 \cdots h_m p_1 \cdots p_n = 1.$$

A group of the above type will be denoted $F(g; \nu_1, \cdots, \nu_k; m; n)$. The elements h_i and p_j are not distinguishable in the abstract group F, but depend on the imbedding of F in $\mathfrak{L}$. The elements h_i are hyperbolic, p_j are parabolic, and these correspond respectively to the boundary curves and punctures in the Riemann surface D/F. These elements h_i, p_j and their conjugates in F will be called *the boundary elements of F*.

We shall say that a finitely generated Fuchsian group F is *finitely maximal* (*f-maximal*) if there does not exist any other Fuchsian group G such that $F \subset G$ and the index $[G: F]$ is finite. We note that if F does not have any hyperbolic boundary elements, then F is f-maximal if and only if there does not exist any other Fuchsian group which contains it. On the other hand, if F does have hyperbolic boundary elements, then there always exist Fuchsian groups G which contain F with infinite index.

By a *geometric isomorphism* (*g-isomorphism*) of a Fuchsian group F, we shall mean an isomorphism $\gamma: F \to \mathfrak{L}$, such that

(1) $\gamma(F)$ is a Fuchsian group.

(2) γ maps the hyperbolic (parabolic) boundary elements of F onto the hyperbolic (parabolic) boundary elements of $\gamma(F)$. Let $\Gamma(F)$ denote the set of g-isomorphisms of F. $\Gamma(F)$ can be topologized in the following way. Let $f_1, \cdots, f_n$ be a set of generators for F. $\Gamma(F)$ can be imbedded in $\mathfrak{L}^n$ by assigning to $\gamma \in \Gamma(F)$ the point $(\gamma(f_1), \cdots, \gamma(f_n)) \in \mathfrak{L}^n$. $\Gamma(F)$ is given the relative topology in $\mathfrak{L}^n$. We introduce an equivalence relation ρ in $\Gamma(F)$. Let $\mathfrak{L}'$ denote the

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group of angle-preserving homeomorphisms of D. ($\mathcal{L}'$ contains orientation reversing transformations. $\mathcal{L}$ is a subgroup of index 2 in $\mathcal{L}'$.) If $\gamma_1, \gamma_2 \in \Gamma(F)$, say that γ_1 is ρ-equivalent to γ_2 if there exists $\lambda \in \mathcal{L}'$ such that

$$\gamma_2(f) = \lambda \gamma_1(f) \lambda^{-1}$$

for all elements $f \in F$. The quotient space

$$T(F) = \Gamma(F)/\rho$$

is then the analogue for F of the Teichmüller space of a Riemann surface. $T(F)$ has been investigated in recent work by L. Ahlfors [**1**; **2**], L. Bers [**3**; **4**; **5**], W. Fenchel and J. Nielsen [**6**] (and further unpublished work by the first two authors). Among other things, it is known that if F is finitely generated, then $T(F)$ is a finite dimensional cell. We shall use the Fenchel-Nielsen theory to establish the results announced in this note.

Let $A(F)$ be the group of g-automorphisms of a Fuchsian group F, and let $I(F)$ be the subgroup of inner automorphisms. The *modular group* $M(F)$ is defined as the quotient group:

$$M(F) = A(F)/I(F).$$

$M(F)$ operates in $T(F)$ in the following way. Let $\alpha \in A(F)$, and consider the map $\Gamma(F) \to \Gamma(F)$ defined by $\gamma \to \gamma \circ \alpha$. As α varies in its $I(F)$-coset, $\gamma \circ \alpha$ varies in its ρ-equivalence class. Therefore, this induces an operation of $M(F)$ on $T(F)$. It is known that $M(F)$ is a properly discontinuous group of transformations of $T(F)$, when F is finitely generated.

2. The results.

THEOREM 1.[2] *Let F and G be finitely generated Fuchsian groups such that $F \subset G$ and the index $[G: F]$ is finite. Let $\iota: F \to G$ denote the injection map. The map $\Gamma(G) \to \Gamma(F)$, defined by $\gamma \to \gamma \circ \iota$ induces a map*

$$m: T(G) \to T(F)$$

which has the following properties.

(1) *m is real analytic and* 1-1.

(2) *The image $I = m[T(G)]$ is a closed subset of $T(F)$.*

(3) *The images of I under the modular group $M(F)$ do not accumulate in $T(F)$.*

[2] The author has been informed that statements (1) and (2) of Theorem 1 are contained in the Ahlfors-Bers theory.

Let $\mathrm{Max}(F)$ denote the set of points in $T(F)$ which represent f-maximal groups. Theorem 1 together with certain area considerations lead to the following.

THEOREM 2. *Let F be a finitely generated Fuchsian group. Then one of the following is true.*

(1) $\mathrm{Max}(F)$ *is empty. There is a group G which contains F with finite index, such that $m[T(G)]=T(F)$ (where m is the map in Theorem 1).*

(2) $\mathrm{Max}(F)$ *is an open, everywhere dense subset of $T(F)$, whose complement is an analytic set.*

Thus, $\mathrm{Max}(F)$ is either the empty set, or it is most of $T(F)$. By some computations which utilize the Fenchel-Nielsen area and modulus formulas, we can actually find all groups F, such that $\mathrm{Max}(F)$ is empty.

THEOREM 3A. *The following groups F are the only finitely generated Fuchsian groups for which $\mathrm{Max}(F)$ is empty.*

(a) *Groups whose limit set consists of two points or less (i.e., cyclic groups and the group $F(0;2,2;1;0)$).*

(b) *Certain triangle groups (which will be enumerated below).*

(c) *The groups F in the following list. Next to each group F, we have listed the unique group G, such that F is a subgroup of finite index in G, and $m[T(G)]=T(F)$. The index $[G:F]$ is always 2.*

F	G
1. $F(0\ ;\ -\ ;\ 1\ ;\ 2)$	$F(0\ ;\ 2\ ;\ 1\ ;\ 0)$
2. $F(0\ ;\ n,n\ ;\ 1\ ;\ 0)$	$F(0\ ;\ 2,n\ ;\ 1\ ;\ 0)$
3. $F(1\ ;\ -\ ;\ 1\ ;\ 0)$	$F(0\ ;\ 2,2,2\ ;\ 1\ ;\ 0)$
4. $F(0\ ;\ -\ ;\ 0\ ;\ 4)$	$F(0\ ;\ 2,2\ ;\ 0\ ;\ 2)$
5. $F(1\ ;\ -\ ;\ 0\ ;\ 2)$	$F(0\ ;\ 2,2,2,2\ ;\ 0\ ;\ 1)$
6. $F(0\ ;\ m,m,n,n\ ;\ 0\ ;\ 0)$	$F(0\ ;\ 2,2,m,n\ ;\ 0\ ;\ 0)$
7. $F(1\ ;\ 2,2\ ;\ 0\ ;\ 0)$	$F(0\ ;\ 2,2,2,2,2\ ;\ 0\ ;\ 0)$
8. $F(2\ ;\ -\ ;\ 0\ ;\ 0)$	$F(0\ ;\ 2,2,2,2,2,2\ ;\ 0\ ;\ 0)$

In case 2 of the above list, $n>2$, and in case 6, $1/m+1/n<1$. In cases 4–5, D/F has finite area, and in cases 6–8, D/F is compact. One would expect case 8 to appear, since every surface of genus 2 is hyperelliptic.

We shall denote the triangle group $F(0;a,b,c;0;0)$ by $T(a,b,c)$, and the parabolic triangle groups $F(0;\, a,\, b;\, 0;\, 1)$, $F(0;\, a;\, 0;\, 2)$ and $F(0;\, -;\, 0;\, 3)$ by $T(a,\, b,\, \infty)$, $T(a,\, \infty,\, \infty)$ and $T(\infty,\, \infty,\, \infty)$, respectively. If F is a triangle group, then F has no deformations, so $T(F)$ consists of a single point. On the other hand, if $T(F)$ consists of a single point, then F is either a triangle group, or an elliptic or parabolic cyclic group. It follows from Theorem 1 that the only groups which can possibly contain a triangle group are other triangle groups.

THEOREM 3B. *The following groups F are the only triangle groups for which* $\mathrm{Max}(F)$ *is empty (i.e., these are the only triangle groups which are not f-maximal).*

(1) *$F=T(m,\, m,\, n)$ is contained in $G=T(2,\, m,\, 2n)$ with index* 2.
(2) *$F=T(2,\, n,\, 2n)$ is contained in $G=T(2,\, 3,\, 2n)$ with index* 3.
(3) *$F=T(3,\, n,\, 3n)$ is contained in $G=T(2,\, 3,\, 3n)$ with index* 4.

In the above theorem, m and n are allowed to take on the value ∞. We remark that the above inclusion relations are not all that occur between triangle groups. There are also the following relations:

(4) $T(\infty,\, \infty,\, \infty)\subset T(3,\, 3,\, \infty)$ with index 3.
(5) $T(\infty,\, \infty,\, \infty)\subset T(2,\, 4,\, \infty)$ with index 4.
(6) $T(4,\, 4,\, 5)\subset T(2,\, 4,\, 5)$ with index 6.
(7) $T(7,\, 7,\, 7)\subset T(2,\, 3,\, 7)$ with index 24.

The above relations 1–7 and those that follow from them are all inclusion relations between triangle groups. We also note that the modular group $T(2,\, 3,\, \infty)$ has not been listed in Theorem 3B, and so it is f-maximal.

Let F be a Fuchsian group of type $F(g;\, -;\, 0;\, 0)$ (i.e., a group which uniformizes a Riemann surface of genus g). Let S be the Riemann surface D/F. If N is the normalizer of F in $\mathfrak{L}$, then the conformal group $C(S)$ is isomorphic to N/F. The previous results imply that for $g>2$, most of the groups isomorphic to F are f-maximal. Thus $N=F$ and $C(S)=\{1\}$ for most Riemann surfaces of genus $g>2$. The following related result can be proved by constructing homomorphisms of Fuchsian groups onto finite groups.

THEOREM 4. *Let G be a nontrivial, finite group. Then there exists a closed Riemann surface S whose conformal group $C(S)$ is isomorphic to G. S may be chosen so that the quotient surface $T=S/C(S)$ has any preassigned genus.*

Let $\mathfrak{F}(S)$ and $\mathfrak{F}(T)$ be the fields of meromorphic function on S and T, respectively. Then $\mathfrak{F}(S)$ is a Galois extension of $\mathfrak{F}(T)$, whose Galois group is isomorphic to G. In particular, if we choose T to be the sphere, then we have found a Galois extension of the field of rational functions, such that the Galois group is isomorphic to the preassigned group G.

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A NOTE ON ENTIRE FUNCTIONS AND A CONJECTURE OF ERDÖS

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1. **Introduction.** Let $f(z)=\sum_0^\infty a_n z^n$ be an entire (transcendental) function and let

$$M(r) = M(r,f) = \max_{|z|=r} |f(z)|, \quad \mu(r) = \mu(r,f) = \max_n (|a_n| r^n).$$

Erdös conjectured that [1] for every entire function, either

$$(1.1) \quad U = U(f) \equiv \limsup_{r\to\infty} \mu(r)/M(r) > u = u(f) \equiv \liminf_{r\to\infty} \mu(r)/M(r),$$

or

$$(1.2) \qquad U(f) = 0.$$

We prove this conjecture, except in one case, when broadly speaking the Taylor series for $f(z)$ has "wide latent" gaps. For $r>0$, let $\nu(r)=\max(n\,|\,\mu(r)=|a_n|r^n)$, and denote by $\{\rho_n\}$ the sequence of jump-

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