MAXIMAL FUCHSIAN GROUPS

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1. DEFINITIONS. Let $D$ be the unit disk $\{z \mid |z| < 1\}$ and let $\mathcal{E}$ be the group of conformal homeomorphisms of $D$. A Fuchsian group is a discrete subgroup of $\mathcal{E}$. We shall be concerned here with the finitely generated Fuchsian groups. It is known that these have the following presentations.

Generators: $a_1, b_1, \ldots, a_g, b_g, e_1, \ldots, e_k, h_1, \ldots, h_m, p_1, \ldots, p_r$.

Defining relations: $e_1^n = e_2^n = \ldots = e_k^n = 1$,

$\prod_{i=1}^{g} a_i b_i a_i^{-1} b_i^{-1} e_1 \cdots e_k h_1 \cdots h_m p_1 \cdots p_n = 1$.

A group of the above type will be denoted $F(g; n_1, \ldots, n_k; m; n)$. The elements $h_i$ and $p_j$ are not distinguishable in the abstract group $F$, but depend on the imbedding of $F$ in $\mathcal{E}$. The elements $h_i$ are hyperbolic, $p_j$ are parabolic, and these correspond respectively to the boundary curves and punctures in the Riemann surface $D/F$. These elements $h_i, p_j$ and their conjugates in $F$ will be called the boundary elements of $F$.

We shall say that a finitely generated Fuchsian group $F$ is finitely maximal (f-maximal) if there does not exist any other Fuchsian group $G$ such that $F \subseteq G$ and the index $[G : F]$ is finite. We note that if $F$ does not have any hyperbolic boundary elements, then $F$ is $f$-maximal if and only if there does not exist any other Fuchsian group which contains it. On the other hand, if $F$ does have hyperbolic boundary elements, then there always exist Fuchsian groups $G$ which contain $F$ with infinite index.

By a geometric isomorphism (g-isomorphism) of a Fuchsian group $F$, we shall mean an isomorphism $\gamma: F \to \mathcal{E}$, such that

(1) $\gamma(F)$ is a Fuchsian group.

(2) $\gamma$ maps the hyperbolic (parabolic) boundary elements of $F$ onto the hyperbolic (parabolic) boundary elements of $\gamma(F)$. Let $\Gamma(F)$ denote the set of $g$-isomorphisms of $F$. $\Gamma(F)$ can be topologized in the following way. Let $f_1, \ldots, f_n$ be a set of generators for $F$. $\Gamma(F)$ can be imbedded in $\mathcal{E}^n$ by assigning to $\gamma \in \Gamma(F)$ the point $\langle \gamma(f_1), \ldots, \gamma(f_n) \rangle \in \mathcal{E}^n$. $\Gamma(F)$ is given the relative topology in $\mathcal{E}^n$. We introduce an equivalence relation $\rho$ in $\Gamma(F)$. Let $\mathcal{E}'$ denote the

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group of angle-preserving homeomorphisms of $D$. ($\mathcal{L}'$ contains orientation reversing transformations. $\mathcal{L}$ is a subgroup of index 2 in $\mathcal{L}'$.) If $\gamma_1, \gamma_2 \in \Gamma(F)$, say that $\gamma_1$ is $\rho$-equivalent to $\gamma_2$ if there exists $\lambda \in \mathcal{L}'$ such that

$$\gamma_2(f) = \lambda \gamma_1(f) \lambda^{-1}$$

for all elements $f \in F$. The quotient space

$$T(F) = \Gamma(F)/\rho$$

is then the analogue for $F$ of the Teichmüller space of a Riemann surface. $T(F)$ has been investigated in recent work by L. Ahlfors [1; 2], L. Bers [3; 4; 5], W. Fenchel and J. Nielsen [6] (and further unpublished work by the first two authors). Among other things, it is known that if $F$ is finitely generated, then $T(F)$ is a finite dimensional cell. We shall use the Fenchel-Nielsen theory to establish the results announced in this note.

Let $A(F)$ be the group of $g$-automorphisms of a Fuchsian group $F$, and let $I(F)$ be the subgroup of inner automorphisms. The modular group $M(F)$ is defined as the quotient group:

$$M(F) = A(F)/I(F).$$

$M(F)$ operates in $T(F)$ in the following way. Let $\alpha \in A(F)$, and consider the map $\Gamma(F) \to \Gamma(F)$ defined by $\gamma \to \gamma \circ \alpha$. As $\alpha$ varies in its $I(F)$-coset, $\gamma \circ \alpha$ varies in its $\rho$-equivalence class. Therefore, this induces an operation of $M(F)$ on $T(F)$. It is known that $M(F)$ is a properly discontinuous group of transformations of $T(F)$, when $F$ is finitely generated.

2. The results.

**Theorem 1.** Let $F$ and $G$ be finitely generated Fuchsian groups such that $F \subset G$ and the index $[G : F]$ is finite. Let $\iota : F \to G$ denote the injection map. The map $\Gamma(G) \to \Gamma(F)$, defined by $\gamma \to \gamma \circ \iota$ induces a map

$$m : T(G) \to T(F)$$

which has the following properties.

1. $m$ is real analytic and 1-1.
2. The image $I = m[T(G)]$ is a closed subset of $T(F)$.
3. The images of $I$ under the modular group $M(F)$ do not accumulate in $T(F)$.

The author has been informed that statements (1) and (2) of Theorem 1 are contained in the Ahlfors-Bers theory.
Let Max(\(F\)) denote the set of points in \(T(F)\) which represent \(f\)-maximal groups. Theorem 1 together with certain area considerations lead to the following.

**Theorem 2.** Let \(F\) be a finitely generated Fuchsian group. Then one of the following is true.

1. Max(\(F\)) is empty. There is a group \(G\) which contains \(F\) with finite index, such that \(m[T(G)] = T(F)\) (where \(m\) is the map in Theorem 1).
2. Max(\(F\)) is an open, everywhere dense subset of \(T(F)\), whose complement is an analytic set.

Thus, Max(\(F\)) is either the empty set, or it is most of \(T(F)\). By some computations which utilize the Fenchel-Nielsen area and modulus formulas, we can actually find all groups \(F\), such that Max(\(F\)) is empty.

**Theorem 3A.** The following groups \(F\) are the only finitely generated Fuchsian groups for which Max(\(F\)) is empty.

(a) Groups whose limit set consists of two points or less (i.e., cyclic groups and the group \(F(0; 2, 2, 1; 0)\)).
(b) Certain triangle groups (which will be enumerated below).
(c) The groups \(F\) in the following list. Next to each group \(F\), we have listed the unique group \(G\), such that \(F\) is a subgroup of finite index in \(G\), and \(m[T(G)] = T(F)\). The index \([G: F]\) is always 2.

<table>
<thead>
<tr>
<th>(F)</th>
<th>(G)</th>
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<tbody>
<tr>
<td>1. (F(0; -; 1; 2))</td>
<td>(F(0; 2; 1; 0))</td>
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<tr>
<td>2. (F(0; n, n; 1; 0))</td>
<td>(F(0; 2, n; 1; 0))</td>
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<tr>
<td>3. (F(1; -; 1; 0))</td>
<td>(F(0; 2,2,2; 1; 0))</td>
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<tr>
<td>4. (F(0; -; 0; 4))</td>
<td>(F(0; 2, 2; 0; 2))</td>
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<tr>
<td>5. (F(1; -; 0; 2))</td>
<td>(F(0; 2,2,2,2; 0; 1))</td>
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<tr>
<td>6. (F(0; m,m,n,n; 0; 0))</td>
<td>(F(0; 2,2,m,n; 0; 0))</td>
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<td>7. (F(1; 2,2; 0; 0))</td>
<td>(F(0; 2,2,2,2,2; 0; 0))</td>
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<tr>
<td>8. (F(2; -; 0; 0))</td>
<td>(F(0; 2,2,2,2,2,2; 0; 0))</td>
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In case 2 of the above list, \(n > 2\), and in case 6, \(1/m + 1/n < 1\). In cases 4–5, \(D/F\) has finite area, and in cases 6–8, \(D/F\) is compact. One would expect case 8 to appear, since every surface of genus 2 is hyperelliptic.

We shall denote the triangle group \(F(0; a, b, c; 0; 0)\) by \(T(a, b, c)\),
and the parabolic triangle groups $F(0; a, b; 0; 1)$, $F(0; a; 0; 2)$ and $F(0; -; 0; 3)$ by $T(a, b, \infty)$, $T(a, \infty, \infty)$ and $T(\infty, \infty, \infty)$, respectively. If $F$ is a triangle group, then $F$ has no deformations, so $T(F)$ consists of a single point. On the other hand, if $T(F)$ consists of a single point, then $F$ is either a triangle group, or an elliptic or parabolic cyclic group. It follows from Theorem 1 that the only groups which can possibly contain a triangle group are other triangle groups.

**Theorem 3B.** The following groups $F$ are the only triangle groups for which Max($F$) is empty (i.e., these are the only triangle groups which are not $f$-maximal).

1. $F = T(m, m, n)$ is contained in $G = T(2, m, 2n)$ with index 2.
2. $F = T(2, n, 2n)$ is contained in $G = T(2, 3, 2n)$ with index 3.
3. $F = T(3, n, 3n)$ is contained in $G = T(2, 3, 3n)$ with index 4.

In the above theorem, $m$ and $n$ are allowed to take on the value $\infty$. We remark that the above inclusion relations are not all that occur between triangle groups. There are also the following relations:

4. $T(\infty, \infty, \infty) \subset T(3, 3, \infty)$ with index 3.
5. $T(\infty, \infty, \infty) \subset T(2, 4, \infty)$ with index 4.
6. $T(4, 4, 5) \subset T(2, 4, 5)$ with index 6.
7. $T(7, 7, 7) \subset T(2, 3, 7)$ with index 24.

The above relations 1–7 and those that follow from them are all inclusion relations between triangle groups. We also note that the modular group $T(2, 3, \infty)$ has not been listed in Theorem 3B, and so it is $f$-maximal.

Let $F$ be a Fuchsian group of type $F(g; -; 0; 0)$ (i.e., a group which uniformizes a Riemann surface of genus $g$). Let $S$ be the Riemann surface $D/F$. If $N$ is the normalizer of $F$ in $\mathcal{L}$, then the conformal group $C(S)$ is isomorphic to $N/F$. The previous results imply that for $g > 2$, most of the groups isomorphic to $F$ are $f$-maximal. Thus $N = F$ and $C(S) = \{1\}$ for most Riemann surfaces of genus $g > 2$. The following related result can be proved by constructing homomorphisms of Fuchsian groups onto finite groups.

**Theorem 4.** Let $G$ be a nontrivial, finite group. Then there exists a closed Riemann surface $S$ whose conformal group $C(S)$ is isomorphic to $G$. $S$ may be chosen so that the quotient surface $T = S/C(S)$ has any pre-assigned genus.

Let $\mathcal{F}(S)$ and $\mathcal{F}(T)$ be the fields of meromorphic function on $S$ and $T$, respectively. Then $\mathcal{F}(S)$ is a Galois extension of $\mathcal{F}(T)$, whose Galois group is isomorphic to $G$. In particular, if we choose $T$ to be the
sphere, then we have found a Galois extension of the field of rational functions, such that the Galois group is isomorphic to the preassigned group $G$.

**REFERENCES**


**Brown University**

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**A NOTE ON ENTIRE FUNCTIONS AND A CONJECTURE OF ERDÖS**

**BY ALFRED GRAY AND S. M. SHAH\(^1\)**

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1. **Introduction.** Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire (transcendental) function and let

$$M(r) = M(r, f) = \max_{|z|=r} |f(z)|, \quad \mu(r) = \mu(r, f) = \max_n (|a_n| r^n).$$

Erdös conjectured that [1] for every entire function, either

$$U = U(f) = \limsup_{r \to \infty} \mu(r)/M(r) > u = u(f) = \liminf_{r \to \infty} \mu(r)/M(r),$$

or

$$U(f) = 0.$$

We prove this conjecture, except in one case, when broadly speaking the Taylor series for $f(z)$ has "wide latent" gaps. For $r > 0$, let $\nu(r) = \max_n (n|\mu(r) = |a_n| r^n)$, and denote by $\{\rho_n\}$ the sequence of jump-

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