sphere, then we have found a Galois extension of the field of rational functions, such that the Galois group is isomorphic to the preassigned group G.

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Brown University

A NOTE ON ENTIRE FUNCTIONS AND A CONJECTURE OF ERDÖS

BY ALFRED GRAY AND S. M. SHAH[1]

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1. **Introduction.** Let $f(z)=\sum_0^\infty a_nz^n$ be an entire (transcendental) function and let

$$M(r) = M(r,f) = \max_{|z|=r} |f(z)|, \quad \mu(r) = \mu(r,f) = \max_n (|a_n| r^n).$$

Erdös conjectured that [1] for every entire function, either

$$(1.1) \quad U = U(f) \equiv \limsup_{r\to\infty} \mu(r)/M(r) > u = u(f) \equiv \liminf_{r\to\infty} \mu(r)/M(r),$$

or

$$(1.2) \qquad U(f) = 0.$$

We prove this conjecture, except in one case, when broadly speaking the Taylor series for $f(z)$ has "wide latent" gaps. For $r>0$, let $\nu(r) = \max(n \mid \mu(r)=|a_n|r^n)$, and denote by $\{\rho_n\}$ the sequence of jump-

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points of $\nu(r)$, so that $0 \leqq \rho_1 \leqq \rho_2 \leqq \cdots$, $\lim_{n\to\infty} \rho_n = \infty$, and $\nu(r) = n$ when $\rho_n \leqq r < \rho_{n+1}$ [2, p. 4]. Let $\{n_k\}$ be the range of $\nu(r)$ for $0 < r < \infty$ and $R = \limsup_{k\to\infty} \{n_{k+1} - n_k\}$, $L = \limsup_{n\to\infty} \rho_{n+1}/\rho_n$.

THEOREM 1.
(1.3) *If* $L > 1$, *then* $U > u$.
(1.4) *If* $L = 1$, $R < \infty$, *then* $U = 0$.
(1.5) *Suppose that* $L = 1$, $R = \infty$ *and*

$$\lim_{k\to\infty} \{\rho_{n_k}/\rho_{n_{k+p}}\}^{n_{k+p} - n_{k+p-1}} = 1,$$

for $p = 1, 2, \cdots$, *then* $U = 0$.

COROLLARY. *If*

$$\liminf_{r\to\infty} \log \mu(r)/(\log r)^2 < \infty \tag{1.6}$$

then $U > u$.

It is not possible to improve on the hypothesis (1.6), for we have

THEOREM 2. *Given any function* $\psi(x)$ *tending to infinity* (*howsoever slowly*) *with* x, *there exists an entire function* $f(z)$, *for which* $U = 0$, *and as* r *tends to infinity*, $\log M(r, f) = o((\log r)^2 \psi(r))$.

2. **Lemma 1.**[2] $u(f) \leqq 2/\pi$.

PROOF. Suppose $|z| = r$ is a value such that at least two terms $a_k z^k$ have moduli equal to $\mu(r)$. If these terms are $a_n z^n$ and $a_m z^m$, then

$$a_n z^n + a_m z^m = \frac{1}{2\pi i} \int_{|\xi|=r} \frac{f(\xi)}{\xi} \left\{ \left(\frac{z}{\xi}\right)^n + \left(\frac{z}{\xi}\right)^m \right\} d\xi.$$

Choose z such that $\arg(a_n z^n) = \arg(a_m z^m)$. Then

$$2\mu(r) \leqq \frac{M(r)}{2\pi} \int_0^{2\pi} \left| 1 + e^{(m-n)i\theta} \right| d\theta = \frac{4M(r)}{\pi}.$$

LEMMA 2. *Let*

$$\liminf_{n\to\infty} \frac{\rho_{n+1}}{\rho_n} = l; \qquad \lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log \mu(r)}{(\log r)^2} = \begin{cases} Q, \\ q. \end{cases}$$

Then

$$1/2 \log L \leqq q \leqq Q \leqq 1/2 \log l.$$

[2] This lemma is due to Dr. J. Clunie. We are thankful to him for communicating this result to one of us.

We omit the proof which is straightforward.

3. **Proof of Theorem 1.** If $P(z)$ is any polynomial, then

$$\mu(r, f + P)/M(r, f + P) \sim \mu(r, f)/M(r, f)$$

and so we may suppose $a_0 = 1$. We have then $0 < \rho_1 \leqq \rho_2 \cdots$. Let

$$F(z) = 1 + \sum_{1}^{\infty} z^n/\rho_1 \cdots \rho_n. \tag{3.1}$$

Then $F(z)$ is an entire function and $M(r, f) \leqq F(r)$, $\mu(r, f) = \mu(r, F)$ for all r. Let $1 < L_1 < L$. There exists a sequence $\{n_p\}$ such that, setting $\rho_n = \rho(n)$,

$$\rho(n_p + 1)/\rho(n_p) > L_1, \qquad p = 1, 2, \cdots. \tag{3.2}$$

Let $z = W\rho(n_p)$. If $1 < |W| < L_1$, then for all p,

$$1 < |W| < \rho(n_p + 1)/\rho(n_p); \ \rho(n_p) < |z| < \rho(n_p + 1). \tag{3.3}$$

Define for these values of z,

$$\mu(z, F) = \mu(z, f) = \mu(W\rho(n_p), f) = (W\rho(n_p))^{n_p}/\rho(1) \cdots \rho(n_p).$$

Then $|\mu(z, f)| = \mu(|z|, f)$, and from (3.1)–(3.3)

$$\frac{F(|z|)}{\mu(|z|, F)} = \frac{F(|W|\rho(n_p))}{\mu(|W|\rho(n_p), F)} \leqq C(W)$$

where

$$C(W) = 1 + \sum_{1}^{\infty} |W|^{-J} + \sum_{1}^{\infty} (|W| L_1^{-1})^J.$$

Define

$$\phi_p(W) = f(W\rho(n_p))/\mu(W\rho(n_p)),$$

and let $\Omega = \{W | 1 < |W| < L_1\}$. For $W \in \Omega$, we have

$$|\phi_p(W)| \leqq M(|W|\rho(n_p), f)/\mu(|W|\rho(n_p), f) \leqq C(W)$$

for all p. Hence $\phi_p(W)$ is analytic in Ω for all p and the family $\{\phi_p(W)\}$ is uniformly bounded on every compact subset of Ω. Hence $\{\phi_p(W)\}$ is a normal family and so there exists a sequence $\{p_k\}$ such that $\{\phi_{p_k}(W)\}$ converges uniformly to a function $G(W)$ on every compact subset of Ω, and $G(W)$ is finite in Ω. Let $1 < R < L_1$. Then $\{\phi_{p_k}(W)\}$ converges uniformly to $G(W)$ on $|W| = R$. Now

$$\left| M(R, \phi_{p_k}) - M(R, G) \right| \leq \max_{|W|=R} \left| \phi_{p_k}(W) - G(W) \right|,$$

and since by uniform convergence

$$\lim_{p_k \to \infty} \max_{|W|=R} \left| \phi_{p_k}(W) - G(W) \right| = 0,$$

we have

$$\lim_{p_k \to \infty} M(R, \phi_{p_k}) = M(R, G).$$

Now

$$M(R, \phi_{p_k}) = \max_{|z| = R\rho(n_{p_k})} \left| \frac{f(z)}{\mu(z)} \right| = \frac{M(R\rho(n_{p_k}), f)}{\mu(R\rho(n_{p_k}), f)}.$$

Hence

$$M(R, G) = \lim_{k \to \infty} M(R\rho(n_{p_k}), f)/\mu(R\rho(n_{p_k}), f).$$

Consider first the case when $G(W)$ is a constant on Ω. Then for $1 < R < L_1$,

$$G(W) = \frac{1}{2\pi i} \int_{|W|=R} \frac{G(W)}{W} dW = \frac{1}{2\pi i} \int_{|W|=R} \left(\lim_{p_k \to \infty} \phi_{p_k}(W)/W \right) dW.$$

By considering the Laurent expansion of $\phi_{p_k}(W)$ about the origin, we obtain

$$1 = \frac{1}{2\pi i} \int_{|W|=R} \{ \phi_{p_k}(W)/W \} dW,$$

and so

$$G(W) = 1 = M(R, G) = \lim_{p_k \to \infty} M(R\rho(n_{p_k}), f)/\mu(R\rho(n_{p_k}), f).$$

Now by Lemma 1, $\limsup_{r \to \infty} M(r, f)/\mu(r, f) \geq \pi/2$ and so $U(f) > u(f)$. If $G(W)$ is not a constant, then let $1 < R_1 < R_2 < R_3 < L_1$. Since $G(W)$ is analytic for $R_1 \leq |W| \leq R_3$, $|G(W)|$ assumes its maximum, for this closed region on either $|W| = R_1$ or $|W| = R_3$ or both. Hence

$$M(R_2, G) < \max\{ M(R_1, G), M(R_3, G) \} = M(R_i, G)$$

say. Then

$$\lim_{k \to \infty} \frac{M(R_i\rho(n_{p_k}), f)}{\mu(R_i\rho(n_{p_k}), f)} \neq \lim_{k \to \infty} \frac{M(R_2\rho(n_{p_k}), f)}{\mu(R_2\rho(n_{p_k}), f)}$$

and so $U(f) > u(f)$ and (1.3) is proved.

To prove (1.4), (1.5) we may assume $a_0 = 1$. Then

$$\rho(1) > 0, \ \rho(n_k) < \rho(n_k + 1) = \cdots = \rho(n_{k+1}) < \cdots, k = 1, 2, \cdots.$$

Further

$$\{M(r, f)\}^2 \geqq \sum_0^\infty |a_n|^2 r^{2n} \geqq 1 + \sum_1^\infty \{r^{n_k}/\rho(1) \cdots \rho(n_k)\}^2.$$

Hence for $\rho(n_k) \leqq r < \rho(n_k + 1)$,

$$\left\{\frac{M(r)}{\mu(r)}\right\}^2 \geqq 1 + \left(\frac{r}{\rho(n_k + 1)}\right)^{2(n_{k+1} - n_k)} + \left(\frac{r}{\rho(n_k + 1)}\right)^{2(n_{k+1} - n_k)} \left(\frac{r}{\rho(n_{k+1} + 1)}\right)^{2(n_{k+2} - n_{k+1})} + \cdots \tag{3.4}$$

and (1.4) follows. To prove (1.5) we note that the second term, third term, $\cdots$ pth term in the right side of (3.4) tend to 1, as $k \to \infty$, and so $U(f) = 0$.

Proof of corollary. By Lemma 2 we must have $L > 1$, and so by (1.3), $U > u$.

The proof of Theorem 2 and the bounds for U and u will be published elsewhere.[3]

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University of Kansas

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