INVARIANT SUBSPACES OF NONSELFADJOINT TRANSFORMATIONS

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This note comments on recent Russian results in Hilbert space. Macaev [9] has given a fundamental estimate of completely continuous transformations which have no nonzero spectrum. The same estimate is true of transformations with imaginary spectrum.

THEOREM I. *Let T be a densely defined transformation in a Hilbert space $\mathcal{H}$ such that T^* has the same domain as T and $T-T^*$ has a completely continuous extension. Suppose that*

$$T - T^* \subset 2i \sum \operatorname{sgn} k \, c_k \bar{c}_k, \tag{1}$$

where (c_k) is an orthogonal set in $\mathcal{H}$, indexed by the odd integers, $\|c_{k+2}\| \leqq \|c_k\|$ for $k>0$, $\|c_{k-2}\| \leqq \|c_k\|$ for $k<0$, and

$$\delta = \sum \|c_k\|^2 / |k| < \infty. \tag{2}$$

If the spectrum of T is imaginary, then the spectrum of $\frac{1}{2}(T+T^*)$ is contained in the interval $[-2\delta/\pi, 2\delta/\pi]$.

If a and b are elements of a Hilbert space, $\bar{a}b$ is the inner product, $\bar{a}b = \langle b, a\rangle$, and $a\bar{b}$ is the linear transformation defined by $(a\bar{b})c = a(\bar{b}c)$ for every c in $\mathcal{H}$. The proof of Theorem I is similar to Macaev's except that it depends on the following new estimate of eigenvalues.

THEOREM II. *Let S be an everywhere defined and bounded transformation in a Hilbert space $\mathcal{H}$, which has imaginary spectrum, such that*

$$S - S^* = 2i \sum b_n \bar{b}_n,$$

where (b_n) is an orthogonal set in $\mathcal{H}$ and $\sum \|b_n\|^2$ is finite. Then,

$$S + S^* = 2 \sum \operatorname{sgn} k \, a_k \bar{a}_k,$$

where (a_k) is an orthogonal set in $\mathcal{H}$, indexed by the odd integers, $\|a_{k+2}\| \leqq \|a_k\|$ for $k>0$, $\|a_{k-2}\| \leqq \|a_k\|$ for $k<0$, and

$$\|a_k\|^2 \leqq (2/\pi)\left(\sum \|b_n\|^2\right)/|k|$$

for every k.

Macaev [9] has given a fundamental existence theorem for invariant subspaces. It can be deduced directly from Theorem I without using, as he indicates, an additional estimate of resolvents. Neither boundedness nor a real spectrum is necessary in the statement of the theorem.

THEOREM III. *Let T be a densely defined transformation in a Hilbert space $\mathcal{H}$ such that T^* has the same domain as T and $T-T^*$ has a completely continuous extension of the form* (1) *where* (2) *holds. If h is a given real number, there exists a closed subspace $\mathcal{M}$ of $\mathcal{H}$, which is invariant under the resolvents of T, such that the restriction of T to $\mathcal{M}$ has its spectrum in the half-plane $x \leqq h$ and the restriction of T^* to the orthogonal complement of $\mathcal{M}$ has its spectrum in the half-plane $x \geqq h$.*

Macaev's existence theorem is stated for transformations which are, in a technical sense, nearly selfadjoint. A similar existence theorem holds for transformations which are nearly unitary.

THEOREM IV. *Let T be an everywhere defined and bounded transformation in a Hilbert space $\mathcal{H}$ which has an everywhere defined and bounded inverse. Suppose that*

$$T^*T - 1 = \sum \epsilon_k c_k \bar{c}_k, \tag{3}$$

where (c_k) is an orthogonal set in $\mathcal{H}$, $\epsilon_k = \pm 1$ for every k, $\|c_{k+1}\| \leqq \|c_k\|$, and

$$\sum \|c_k\|^2 / |k| < \infty. \tag{4}$$

If α is a given real number, $0 < \alpha < \pi$, then there exists a closed subspace $\mathcal{M}$ of $\mathcal{H}$ which is invariant under T and T^{-1}, such that the restriction of T to $\mathcal{M}$ has its spectrum in the sector $-\alpha \leqq \theta \leqq \alpha$, and the restriction of T^* to the orthogonal complement of $\mathcal{M}$ has its spectrum in the complementary sector $\alpha \leqq \theta \leqq 2\pi - \alpha$.

Invariant subspaces of this nature need not exist if the hypotheses of Theorem IV are not satisfied.

THEOREM V. *Let (c_k) be an orthogonal set in a Hilbert space $\mathcal{H}$ such that $\|c_{k+1}\| \leqq \|c_k\| < 1$ for every k,* $\lim c_k = 0$, *and* (4) *is not satisfied. Let $\epsilon_k = \pm 1$ for every k. Then there exists an everywhere defined and bounded transformation T in $\mathcal{H}$, with an everywhere defined and bounded inverse, which satisfies* (3), *and the spectrum of the restriction of T to every nonzero closed subspace invariant under T and T^{-1} is the full unit circle $|z| = 1$.*

The proof of Theorem V depends on the theory of translation invariance. If $W(x)$ is a complex valued function of integral x, consider the corresponding Hilbert space of functions $f(x)$ of integral x, such that

$$\|f\|^2 = \sum |f(n)/W(n)|^2 < \infty.$$

If $W(x)/W(x-1)$ and $W(x)/W(x+1)$ are bounded, the translation operator $T: f(x) \to f(x-1)$ is bounded and has a bounded inverse. Complete continuity of T^*T-1 means that

$$\lim \left| W(x)/W(x-1) \right| = 1$$

as $|x| \to \infty$, and in this case the spectrum of the transformation is the unit circle. If $|W(x)|$ is increasing for negative x and is decreasing for positive x, condition (4) is equivalent to

$$\sum (1 + n^2)^{-1} \log \left| W(n) \right| > -\infty.$$

The proof of Theorem V is completed using a theorem of Levinson, as it is stated in [1].

In the situation of Theorem III, T has an integral representation of the form

$$T = \int h(t)dP(t) + \int P(t)(T - T^*)dP(t),$$

where $P(x)$ is a nondecreasing function whose values are projections into invariant subspaces for the resolvents of T. The first term on the right is a selfadjoint transformation. The second term is an everywhere defined and bounded transformation with imaginary spectrum. The theory of this second integral is that of Gohberg and Kreĭn [6], except that it is not restricted to transformations which have the origin as the only point in the spectrum. The integration theory involves three distinct topics: (a) the uniqueness of transformations with given invariant subspaces, (b) the existence of sufficiently many invariant subspaces to characterize a given transformation, and (c) the existence of transformations with given invariant subspaces. Theorem I is the essential estimate in each case.

A fundamental problem is to determine the uniqueness of such integral representations. The essential difficulty is due to the lack of information about invariant subspaces of transformations whose spectrum is a point. In special cases the invariant subspaces are totally ordered by inclusion. Results of this nature are obtainable from the theory of Hilbert spaces of entire functions [2]. This theory contains implicitly a determination of the invariant subspaces of transformations T, with no nonzero spectrum, when $T-T^*$ has two dimensional range and its eigenvalues are not on the same side of the real axis. See [3] for the relation between Hilbert spaces of entire functions and invariant subspaces of transformations. In particular the results of [2] may be used to verify a conjecture of Kreĭn, stated by Brodskiĭ [5], that the real invariant subspaces of the above transformations are totally ordered by inclusion.

Unfortunately Theorems VI and VIII of [3] are erroneous as stated. Theorem VIII is easily corrected, but we can find no valid form of Theorem VI which does not leave a gap between the problem of invariant subspaces and factorization problems for operator valued analytic functions. What is false in Theorem VI is that $M(a, b, z)$, $M(b, c, z)$, and $M(a, c, z)$ need satisfy condition (4) there, which implies that the corresponding spaces have a trivial structure. As a result the existence of invariant subspaces is not known in all cases in which the $M(z)$ function can be factored.

Added in proof. The following hypothesis should be added to Theorem V. The orthogonal set (c_k) is complete in $\mathcal{H}$ unless there are only a finite number of positive ϵ_k or of negative ϵ_k, in which case the orthogonal complement of the c_k is of countably infinite dimension.

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