BOUNDED APPROXIMATION BY POLYNOMIALS[1]

BY L. A. RUBEL AND A. L. SHIELDS

Communicated by Maurice Heins, April 22, 1963.

We announce a complete solution to the following problem. If G is an arbitrary bounded open set in the complex plane, which complex-valued functions in G can be obtained as the bounded pointwise limits in G of a sequence of polynomials?

THEOREM. *Given an arbitrary bounded open set G in the complex plane, and a complex-valued function f defined on G. There exists a sequence $\{p_n\}$ of polynomials that are uniformly bounded on G and that converge pointwise on G to f if and only if f has an extension F that is bounded and holomorphic on G^*, where G^* is the inside of the outer boundary of G.*

More precisely, G^* is the complement of the closure of the unbounded component of the complement of the closure of G.

In a certain sense, this result lies somewhere between Runge's theorem and Mergelyan's theorem [3]. The correct formulation of our theorem is in terms of sequences, and not topological closure. Indeed, for a certain bounded open set G, there exists a function f and functions f_n such that (i) each f_n is the bounded limit of a sequence of polynomials, (ii) f is the bounded limit of f_n, but (iii) f is not the bounded limit of any sequence of polynomials.

With G and G^* as above, we write $B_H(G)$ for the set of bounded holomorphic functions on G, and $B_H(G^*: G)$ for the set of functions on G that have a bounded holomorphic extension to G^*, and $P(G)$ for the set of functions on G that can be boundedly approximated on G by a sequence of polynomials. The theorem now reads:

$$P(G) = B_H(G^*: G).$$

Even if G is connected and simply connected, it may happen that G^* has several components. We define $G^\#$ as the union of those components of G^* that intersect G. Clearly, $B_H(G^*: G) = B_H(G^\#: G)$.

As a corollary to the theorem, we get a characterization of those bounded open sets G on which each bounded holomorphic function can be boundedly approximated by polynomials, namely $P(G) = B_H(G)$ if and only if $B_H(G) = B_H(G^*: G)$; in other words, if and only if the inner boundary of G is a set of removable singularities for all bounded holomorphic functions on G. The inner boundary of

[1] This research was partially supported by the National Science Foundation.

G is that part of the boundary of G that does not intersect the unbounded component of the complement of G closure.

For example, if G is the open unit disc, then $G^* = G$, and hence $P(G) = B_H(G)$. This is a classical result; in this case, the sequence of arithmetic means of the partial sums of the power series of f converges boundedly to f. A proof that $P(G) \supseteq B_H(G)$ if G is a Jordan domain is given in [**2**, pp. 3–5]. If G is the punctured unit disc, then $P(G) = B_H(G)$, but if G is the unit disc with a radius removed, then $P(G) \neq B_H(G)$. It may happen that $G = G^\#$, and consequently $P(G) = B_H(G)$, even though the closure of G separates the plane. This is the case if G is a spiral ribbon, with infinitely many turns, that winds down onto the unit disc.

The proof of the theorem is long, with complications arising if G has infinitely many components. We give here a crude sketch of a line of proof that works in the special case where G is connected, which is considerably easier than the general case. We must show that $P(G) = B_H(G^\# : G)$.

First, $P(G) \subseteq B_H(G^\# : G)$, since if the sequence $\{p_n\}$ of polynomials converges boundedly to f, then the p_n are uniformly bounded on the outer boundary of G, hence on the boundary of $G^\#$ also, and the p_n are consequently uniformly bounded in $G^\#$. Hence, some subsequence converges on $G^\#$ to a bounded holomorphic function which must be an extension of f.

In the other direction, there exist simply connected regions G_n, with $G_{n+1} \subseteq G_n$, and the closure of $G^\#$ contained in each G_n. The regions G_n "squeeze down" onto $G^\#$ in the sense that $G^\#$ is the largest connected open superset of G that is contained in $\bigcap G_n$. We construct the regions G_n as the insides of a sequence of equipotential curves of the logarithmic equilibrium potential on G. By a theorem of Carathéodory [**1**, p. 76], if ϕ_n is the normalized mapping function of G_n onto the unit disc, then the ϕ_n converge to ϕ on $G^\#$, where ϕ is the normalized mapping function of $G^\#$. Given f in $B_H(G^\#)$, the functions $f_n = f \circ \phi^{-1} \circ \phi_n$ converge boundedly in $G^\#$ to f. By Runge's theorem, f_n can be approximated on $G^\#$ by a polynomial p_n, with a uniform error at most $1/n$. The polynomials p_n converge boundedly to f in $G^\#$, and a fortiori in G. Thus $B_H(G^\# : G) \subseteq P(G)$. This completes the sketch of the proof for the special case.

References

1. C. Carathéodory, *Conformal representation*, Cambridge Univ. Press, Cambridge, 1958.

2. T. Carleman, *Über die Approximation analytischer Funktionen durch lineare*

Aggregate von vorgegebenen Potenzen, Arkiv för Matematik, Astronomi och Fysik **17** (1922–1923), no. 9, 1–30.

3. S. N. Mergelyan, *Uniform approximation to functions of a complex variable*, Amer. Math. Soc. Transl. **101** (1954), 2–99.

COLUMBIA UNIVERSITY,
UNIVERSITY OF ILLINOIS, AND
UNIVERSITY OF MICHIGAN

DOUBLY INVARIANT SUBSPACES OF ANNULUS OPERATORS

BY DONALD SARASON[1]

Communicated by P. R. Halmos, April 29, 1963

1. Introduction. Let C be the unit circle in the complex plane and let C_0 be the circle $\{z: |z| = r_0\}$, where r_0 is a positive real number less than unity. The set $C \cup C_0$ is the boundary of the annulus $A = \{z: r_0 < |z| < 1\}$. Let us endow the circles C and C_0 with Lebesgue measure of total mass unity, and denote by $L^2(\partial A)$ the L^2 space associated with the measure thereby defined on the set $C \cup C_0$. This note concerns the invariant subspaces of the position operator on the space $L^2(\partial A)$, that is, of the operator Z on $L^2(\partial A)$ defined by $(Zx)(z) = zx(z)$.

We may regard $L^2(\partial A)$ as the direct sum of the two spaces $L^2(C)$ and $L^2(C_0)$. As subspaces of $L^2(\partial A)$, the latter reduce the operator Z. The restriction of Z to $L^2(C)$ is a well-known operator, a so-called bilateral shift (of unit multiplicity). The invariant subspaces of this operator have been extensively studied by Beurling [**1**], by Helson and Lowdenslager [**3**], and by Halmos [**2**]. The restriction of Z to $L^2(C_0)$ is a bilateral shift multiplied by the scalar r_0, and so has the same invariant subspace structure as a bilateral shift. The operator Z is therefore the direct sum of two operators whose invariant subspaces have been completely described. However, the problem of determining the invariant subspaces of Z involves more than merely a routine extension of known results about bilateral shifts, and as yet has not been solved completely.

---

[1] Research supported in part by the National Science Foundation. The results announced in this paper constitute a portion of the author's University of Michigan Doctoral Dissertation. I am deeply indebted to Professor Paul Halmos for the help he has given me over the past year.