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3. S. N. Mergelyan, *Uniform approximation to functions of a complex variable*, Amer. Math. Soc. Transl. 101 (1954), 2–99.

COLUMBIA UNIVERSITY,
UNIVERSITY OF ILLINOIS, AND
UNIVERSITY OF MICHIGAN

DOUBLY INVARIANT SUBSPACES OF ANNULUS OPERATORS

BY DONALD SARASON¹

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1. **Introduction.** Let C be the unit circle in the complex plane and let C_0 be the circle $\{z: |z| = r_0\}$, where r_0 is a positive real number less than unity. The set $C \cup C_0$ is the boundary of the annulus $A = \{z: r_0 < |z| < 1\}$. Let us endow the circles C and C_0 with Lebesgue measure of total mass unity, and denote by $L^2(\partial A)$ the L^2 space associated with the measure thereby defined on the set $C \cup C_0$. This note concerns the invariant subspaces of the position operator on the space $L^2(\partial A)$, that is, of the operator Z on $L^2(\partial A)$ defined by $(Zx)(z) = zx(z)$.

We may regard $L^2(\partial A)$ as the direct sum of the two spaces $L^2(C)$ and $L^2(C_0)$. As subspaces of $L^2(\partial A)$, the latter reduce the operator Z . The restriction of Z to $L^2(C)$ is a well-known operator, a so-called bilateral shift (of unit multiplicity). The invariant subspaces of this operator have been extensively studied by Beurling [1], by Helson and Lowdenslager [3], and by Halmos [2]. The restriction of Z to $L^2(C_0)$ is a bilateral shift multiplied by the scalar r_0 , and so has the same invariant subspace structure as a bilateral shift. The operator Z is therefore the direct sum of two operators whose invariant subspaces have been completely described. However, the problem of determining the invariant subspaces of Z involves more than merely a routine extension of known results about bilateral shifts, and as yet has not been solved completely.

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The purpose of this note is to announce results concerning the invariant subspaces of Z that are also invariant under Z^{-1} ; these we call *doubly invariant subspaces* of Z . The proofs, to appear elsewhere, depend to a large extent on complex function theory, and involve analogues for functions in the H^2 space of the annulus A of a number of well-known properties of functions in the H^2 space of a disk.

2. A characterization of doubly invariant subspaces. For every real number α we define the function w_α on $C \cup C_0$ by

$$\left. \begin{aligned} w_\alpha(e^{it}) &= e^{i\alpha t} \\ w_\alpha(r_0 e^{it}) &= r_0^\alpha e^{i\alpha t} \end{aligned} \right\} 0 \leq t < 2\pi.$$

Then $Zw_\alpha = w_{\alpha+1}$, and the functions w_α and $w_{\alpha+n}$ are orthogonal for $n = \pm 1, \pm 2, \dots$. For $0 \leq \alpha < 1$ we denote by $H_\alpha^2(\partial A)$ the smallest doubly invariant subspace of Z containing w_α , that is, the span in $L^2(\partial A)$ of the functions $w_{\alpha+n}$, $n = 0, \pm 1, \pm 2, \dots$. More generally, for any function x in $L^2(\partial A)$ we let M_x denote the smallest doubly invariant subspace of Z containing x . The subspaces $H_\alpha^2(\partial A)$ are prototypes of the doubly invariant subspaces that do not reduce Z . More precisely, we have the following two theorems.

THEOREM 1. *Let x be a function in $L^2(\partial A)$. If the condition*

$$(*) \quad \int_0^{2\pi} \log |x(e^{it})| dt + \int_0^{2\pi} \log |x(r_0 e^{it})| dt > -\infty$$

is satisfied, let α be the number in the interval $[0, 1)$ congruent modulo 1 to the number

$$\frac{1}{2\pi q_0} \left[\int_0^{2\pi} \log |x(e^{it})| dt - \int_0^{2\pi} \log |x(r_0 e^{it})| dt \right],$$

where $q_0 = -\log r_0$. Then there is a measurable function w on $C \cup C_0$, with $|w| = 1$ almost everywhere, such that M_x consists of all products wy with y in $H_\alpha^2(\partial A)$. The function w is unique to within a multiplicative constant of unit modulus.

On the other hand, if condition () is not satisfied (that is, if the function x is "small"), then M_x consists of all functions in $L^2(\partial A)$ that vanish at every point where x vanishes.*

THEOREM 2. *If M is any doubly invariant subspace of Z , then there is a function x such that $M = M_x$.*

Theorems 1 and 2 characterize the doubly invariant subspaces of Z . We see in particular that if x is a function in $L^2(\partial A)$ satisfying (*),

and if α is as defined in Theorem 1, then the operator $Z|M_\alpha$ is unitarily equivalent to $Z|H^2_\alpha(\partial A)$. On the other hand, one can show that the operators $Z|H^2_\alpha(\partial A)$ and $Z|H^2_\beta(\partial A)$ are not unitarily equivalent for $\alpha \neq \beta$.

3. **Doubly invariant subspaces in $H^2(A)$.** The space $H^2(A)$ consists by definition of all holomorphic functions f in the annulus A such that

$$\sup_{r_0 < r < 1} \int_0^{2\pi} |f(re^{it})|^2 dt < \infty.$$

Just as in a disk, a function of class $H^2(A)$ has nontangential limits at almost every boundary point of A , and so can be extended (almost everywhere) to the boundary. By Fatou's lemma, the resulting boundary function belongs to $L^2(\partial A)$. In fact, one can show that the boundary function belongs to $H^2_0(\partial A)$. Conversely, any function in $H^2_0(\partial A)$ is the boundary function of a unique function in $H^2(A)$. The spaces $H^2_0(\partial A)$ and $H^2(A)$ are thus in one-to-one correspondence, and the latter is thereby endowed with a Hilbert space structure. The operator $Z|H^2_0(\partial A)$ corresponds to the operator Z_0 on $H^2(A)$ defined by $(Z_0f)(z) = zf(z)$. We conclude with two results concerning doubly invariant subspaces of the operator Z_0 .

THEOREM 3. *Let a_1, a_2, a_3, \dots be a finite or infinite sequence of points in the annulus A (repetitions allowed). Let M be the collection of all functions in $H^2(A)$ that vanish (with the appropriate multiplicity) at each point a_k . Then M is a doubly invariant subspace of Z_0 . If the sum*

$$(**) \quad \sum \min\left(1 - |a_k|, 1 - \frac{r_0}{|a_k|}\right)$$

*is infinite, then M is trivial. If the sum $(**)$ is finite, then $Z_0|M$ is unitarily equivalent to $Z|H^2_\alpha(\partial A)$, where α is the number in the interval $[0, 1)$ congruent modulo 1 to the number $\sum \alpha_k$, the α_k being defined by*

$$\alpha_k = \begin{cases} \frac{-1}{q_0} \log |a_k| & \text{if } r_0^{1/2} \leq |a_k| < 1, \\ \frac{-1}{q_0} \log (|a_k|/r_0) & \text{if } r_0 < |a_k| < r_0^{1/2}, \end{cases}$$

$$q_0 = -\log r_0.$$

We shall say that a function in $H^2(A)$ is a cyclic vector of Z_0 if it is contained in no proper doubly invariant subspace of Z_0 .

THEOREM 4. *A function $f \neq 0$ in $H^2(A)$ is a cyclic vector of Z_0 if and only if it satisfies the condition*

$$\int_0^{2\pi} \log |f(r_0^\delta e^{it})| dt = \delta \int_0^{2\pi} \log |f(r_0 e^{it})| dt \\ + (1 - \delta) \int_0^{2\pi} \log |f(e^{it})| dt \quad \text{for } 0 < \delta < 1.$$

REFERENCES

1. A. Beurling, *On two problems concerning linear transformations in Hilbert space*, Acta Math. **81** (1949), 239–255.
2. P. R. Halmos, *Shifts on Hilbert spaces*, J. Reine Angew. Math. **208** (1961), 102–112.
3. H. Helson and D. Lowdenslager, *Invariant subspaces*, Proceedings of the International Symposium on Linear Spaces, Jerusalem Academic Press, 1961, pp. 251–265.

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