RESEARCH ANNOUNCEMENTS

The purpose of this department is to provide early announcement of significant new results, with some indications of proof. Although ordinarily a research announcement should be a brief summary of a paper to be published in full elsewhere, papers giving complete proofs of results of exceptional interest are also solicited.

TWO THEOREMS ON NONLINEAR FUNCTIONAL EQUATIONS IN HILBERT SPACE

BY GEORGE J. MINTY

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Let H be a Hilbert space, with real or complex scalars. A function $F: H \to H$ is called *monotonic* provided, for any $x_1, x_2 \in H$, we have $\mathrm{Re}\langle x_1 - x_2, Fx_1 - Fx_2 \rangle \geqq 0$. If ($\geqq$) is replaced by ($>$), it is *strictly* monotonic, and if 0 is replaced by $c\|x_1 - x_2\|^2$, with $c > 0$, it is *strongly* monotonic. Examples are: the gradient of a convex (resp. strictly or strongly convex) function, the negative of a linear dissipative operator, a linear operator satisfying $\mathrm{Re}\langle x, Fx \rangle \geqq c\|x\|^2$ (the hypothesis of a form of the Lax-Milgram Lemma), and so on.

A variant, due to F. E. Browder, of a theorem of the author [5, Corollary to Theorem 4] asserts that a continuous, everywhere-defined strongly monotonic function has a continuous everywhere-defined inverse. (Browder has also generalized the theorem.) These results are used in the proofs of the following theorems:

THEOREM 1. *If F is everywhere-defined, continuous, and monotonic, and satisfies for some real M*

$$\|x\| > M \text{ implies } \mathrm{Re}\,\langle x, Fx \rangle \geqq 0 \tag{1}$$

then the equation $Fx = \theta$ has a solution, which is unique if F is strictly monotonic.

THEOREM 2. *If K and F are everywhere-defined, continuous, and monotonic, K is linear, and in addition F is a bounded operator and satisfies* (1), *then the "Hammerstein equation" $x + KFx = \theta$ has a solution; the solution is unique if either K or F is strictly monotonic.*

A (nonlinear) operator is called "bounded" if it maps bounded sets into bounded sets.

A VARIANT ON THEOREM 2. *If K is strongly monotonic, the hypotheses of boundedness of F can be dropped from Theorem 2.*

The proofs will appear in [2]. The application of Theorem 2 to nonlinear integral equations generalizes results of E. H. Rothe [6] valid for self-adjoint K. Theorem 1 appears to be a useful tool for the study of nonlinear differential equations.

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University of Michigan