THE GEOMETRY OF IMMERSIONS. I

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We hereby announce some general methods for higher order differential geometry. Our main tools are a generalization of the general position theory of Whitney and Thom, and a characteristic class theory for higher order bundles having given higher order connections.

The second part of this announcement will deal with applications of this machinery to some problems of geometric singularities. One application will be to count the number of umbilic points on an immersed hypersurface. Full details will appear in a separate publication.

1. ***p*th order osculating maps.** It is known [4] that on each smooth manifold X a sequence of smooth vector bundles $\{T_k(X)\}_{k=1,2,\cdots}$ over X can be canonically constructed. $T_p(X)$ is called the *bundle of pth order tangent vectors over* X, and $T_1(X)$ is just the tangent bundle of X. These bundles furthermore satisfy short exact sequences

$$0 \to T_{p-1}(X) \to T_p(X) \to 0^p T_1(X) \to 0$$

where $0^p T_1(X)$ denotes the p-fold symmetric tensor product of the tangent bundle. It is also known that to each smooth map f between manifolds X and Y there is canonically defined a *pth order differential* $T_p(f): T_p(X) \to T_p(Y)$ which is a homomorphism of smooth vector bundles covering f. For each smooth f there is the following family of commutative diagrams of vector bundles with exact rows,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T_{p-1}(X) & \longrightarrow & T_p(X) & \longrightarrow & 0^p T_1(X) & \longrightarrow & 0 \\ & & \downarrow T_{p-1}(f) & & \downarrow T_p(f) & & \downarrow 0^p T_1(f) & & \\ 0 & \longrightarrow & T_{p-1}(Y) & \longrightarrow & T_p(Y) & \longrightarrow & 0^p T_1(Y) & \longrightarrow & 0. \end{array}$$

If the dimension of X is n, then the fiber dimension of $T_p(X)$ is

$$\nu(n, p) = n + \binom{n+1}{2} + \cdots + \binom{n+p-1}{p}.$$

The smooth sections $S(T_p(X))$ of $T_p(X)$ will be called the *pth order*

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vector fields. They are just the qth order linear partial differential operators generated by the vector fields $q \leqq p$.

DEFINITION. A *pth order* ($p > 1$) *symmetric linear connection* on a manifold X is a linear connection D on $T_p(X)$ such that:

(1) $D_v T = v \cdot T$, $v \in S(T(X))$, $T \in S(T_p(X))$ and $\cdot$ function composition,

(2) $R \mid S(T_{p-1}(X)) = 0$ where R is the curvature tensor of the connection D. For $p = 1$ we will use the usual notion of a symmetric linear connection. Then the following generalization of a well-known classical theorem [**1**; **2**] can be proven.

PROPOSITION 1.1. *Let X be a smooth manifold. The set of pth order symmetric linear connections on X are in a* 1-1 *correspondence with smooth splittings of the short exact sequence*

$$0 \to T_p(X) \to T_{p+1}(X) \to 0^{p+1} T_1(X) \to 0.$$

Finally, if X is a Euclidean space there exists a canonical set of splittings of the above exact sequences given by the linear structure on X.

We will now fix the following data and notation for the next few sections. Let X and Y be manifolds of dimension n and N respectively. Let $C(X, Y)$ denote the space of C^∞ functions from X to Y with a sufficiently strong topology. Let $I(X, Y)$ and $E(X, Y)$ be the spaces of immersions and embeddings of X in Y respectively. $I(X, Y)$ and $E(X, Y)$ are known to be open subsets of $C(X, Y)$. Let $(D_k)_{k=1,2,3,\ldots}$ be a sequence of kth order symmetric linear connections on Y. Denote by the same letter D_k the map from $T_{k+1}(Y) \to T_k(Y)$ which splits the exact sequence

$$0 \to T_k(Y) \to T_{k+1}(Y) \to 0^{k+1}(Y) \to 0$$

induced by D_k.

DEFINITION. Let $f \in C(X, Y)$. Let $\nabla_p T_p(f)$: $T_p(X) \to T_1(Y)$ be the smooth vector bundle homomorphism covering f defined by $D_1 D_2 \cdots D_{p-1} T_p(f)$. $\nabla_p T_p(f)$ is called the *pth order osculating map of f* with respect to the connections $(D_k)_{k=1,2,\ldots}$.

REMARK. Explicit reference to the connections will be dropped whenever no confusion can arise.

2. Transversality theorems.

DEFINITION. Let $\xi = (\pi: E \to B)$ be a smooth fiber bundle with fiber F. The pair of spaces (K, K') is said to be an *admissible submanifold of E* if

(1) K is a submanifold of E,

(2) K' is a submanifold of F, and

(3) there exists an atlas $\mathfrak{A} = \{(U, \phi_U)\}$ of E such that $\phi_U(\pi^{-1}(U) \cap K) = U \times K'$ for all $(U, \phi_U) \in \mathfrak{A}$.

DEFINITION. Let $\xi = (\pi: E \to B)$ and $\xi' = (\pi': E' \to B')$ be vector bundles. Denote by $\mathrm{Hom}(E, E')$ the vector bundle over $B \times B'$ with fiber $\mathrm{Hom}(E_x, E'_y)$ over $(x, y) \in B \times B'$ induced in the obvious way from ξ and ξ'.

Let E be a smooth vector sub-bundle of $T_p(X)$. Let $f \in C(X, Y)$. We then define a map $\Delta_E(f): X \to \mathrm{Hom}(E, T_1(Y))$, which is a section over the graph of f defined by $\Delta_E(f)(x) = \nabla_p T_p(f) | E_x \in \mathrm{Hom}(E_x, T_1(Y)_{f(x)})$.

THEOREM 2.1. *Let (K_i, K'_i) be a finite set of disjoint admissible submanifolds of* $\mathrm{Hom}(E, T_1(Y))$. *Let $A = (f \in C(X, Y) | \Delta_E(f)(X)$ meets K_i transversally for all i). Then A is open and dense in $C(X, Y)$.*

THEOREM 2.2. *Let F be a closed subset of X such that $X - F$ is a smooth manifold. Let E be a vector sub-bundle of $T_p(X)$. Let (K_i, K'_i) be a finite set of disjoint admissible submanifolds of* $\mathrm{Hom}(E, T_1(Y))$. *Let $f_0 \in C(X, Y)$ such that $\Delta_E(f_0)$ is transversal to all the K_i on F. Let $C_F(f_0) = (g \in C(X, Y): g | F = f_0 | F)$. Then the set of $g \in C_F(f_0)$ such that $\Delta_E(g)(X)$ meets the K_i transversally for all i is dense in $C_F(f_0)$.*

3. **Nondegenerate immersions and homotopies.** Let $\nu(n, p)$ denote the fiber dimension of $T_p(X)$ throughout this note.

DEFINITION. Let $f \in C(X, Y)$. f is said to be *pth order nondegenerate* if the pth order osculating map $\nabla_p T_p(f): T_p(X) \to T_1(Y)$ is of maximal rank on each fiber.

REMARK. If $N \geqq \nu(n, p)$ then pth order nondegenerate maps are kth order nondegenerate for $k \leqq p$. Hence they are immersions.

THEOREM 3.1. *If $N \leqq \nu(n, p) - n$ or if $N \geqq \nu(n, p) + n$ the set of pth order nondegenerate maps are open and dense in $C(X, Y)$.*

COROLLARY I. *If $n < N \leqq \nu(n, p) - n$, the set of pth order nondegenerate immersions (embeddings) are open and dense in $I(X, Y)$ ($E(X, Y)$).*

COROLLARY II. *Let $F \subseteq X$ be a closed subset such that $X - F$ is a manifold. Let $f_0 \in C(X, Y)$ be a map which is pth order nondegenerate on F. Let $C_F(f_0) = (g \in C(X, Y): g | F = f_0 | F)$. If $N \leqq \nu(n, p) - n$ or if $N \geqq \nu(n, p) + n$, then the set of $g \in C_F(f_0)$ which are pth order nondegenerate on X, is dense in $C_F(f_0)$.*

Let $I = [0, 1]$ be the closed unit interval.

DEFINITION. Let $f_0, f_1 \in I(X, Y)$ ($E(X, Y)$). Assume that f_0 and f_1 are pth order nondegenerate. A smooth homotopy $F: I \times X \to Y$ is called a *pth order regular homotopy* (*pth order isotopy*) between f_0 and f_1 if

(a) $F|X\times\{0\}=f_0$,

(b) $F|X\times\{1\}=f_1$, and

(c) $F|X\times\{t\}=f_t$ is a pth order nondegenerate immersion (embedding) for every $t\in I$.

THEOREM 3.2. *Let f_0 and f_1 be a pth order nondegenerate immersion of X in Y.*

(a) *If $N\geqq\nu(n, p)+n+1$, and if f_0 and f_1 are homotopic, then they are pth order regularly homotopic.*

(b) *Let f_0 and f_1 also be embeddings. If*

$$N \geqq \max(\nu(n, p) + n + 1, 2n + 3)$$

and if f_0 and f_1 are homotopic, then they are pth order isotopic.

(c) *If $n<N\leqq\nu(n, p)-n-1$, and if f_0 is regularly homotopic to f_1, then they are pth order regularly homotopic.*

(d) *If $n<N\leqq\nu(n, p)-n-1$, if f_0 and f_1 are isotopic embeddings, then they are pth order isotopic embeddings.*

The theorems of this section are proved by using the general theorems stated in the last section.

4. pth order normal bundles and conormal bundles. *N.B.* Let $\xi=(\pi: E\rightarrow B)$ and $\xi'=(\pi': E'\rightarrow B')$ be vector bundles. Let $h: E\rightarrow E'$ be a vector bundle homomorphism covering $f: B\rightarrow B'$. Denote by $h!$ the vector bundle homomorphism from E to $f^{-1}E'$ over B, canonically induced from h.

DEFINITION. Let $f: X\rightarrow Y$ be a pth order nondegenerate immersion.

(a) Let $N>\nu(n, p)$. Then $\nabla_pT_p(f)!T_p(X)\rightarrow f^{-1}T_1(Y)$ is a monomorphism on each fiber. Let $N_{p,f,Y}(X)$ = cokernel$(\nabla_pT_p(f)!)$. $N_{p,f,Y}(X)$ is called the pth *order normal bundle of X in Y with respect to f*. If $Y=R^N$, denote $N_{p,f,Y}(X)$ by $N_{p,f}(X)$ and call it the *pth order normal bundle of X with respect to f*.

(b) Let $N<\nu(n,p)$. $\nabla_pT_p(f)!$: $T_p(X)\rightarrow f^{-1}T_1(Y)$ is an epimorphism of vector bundles over X. Let $K_{p,f,Y}(X)$ = kernel$(\nabla_pT_p(f)!)$. $K_{p,f,Y}(X)$ is called the *pth order conormal bundle of X in Y with respect to f*. Similarly if $Y=R^N$, $K_{p,f,Y}(X)$ is denoted by $K_{p,f}(X)$ and is called the *pth order conormal bundle of X with respect to f*.

PROPOSITION. *Let $f: X\rightarrow Y$ be a pth order nondegenerate immersion.*

(a) *If $N>\nu(n, p)$ then $f^{-1}T_1(Y)\cong T_p(X)\oplus N_{p,f,Y}(X)$.*

(b) *If $N<\nu(n, p)$ then $T_p(X)\cong f^{-1}T_1(Y)\oplus K_{p,f,Y}(X)$.*

COROLLARY. *Furthermore assume that* $Y=R^N$.

(a) *If* $N>\nu(n,p)$ *then* $X\times R^N\cong T_p(X)\oplus N_{p,f}(X)$.

(b) *If* $N<\nu(n,p)$ *then* $T_p(X)\cong X\times R^N\oplus K_{p,f}(X)$.

REMARK. Let $Y=R^N$, and let $N>\nu(n,p)$. Let $f: X\to R^N$ be a pth order nondegenerate immersion. Then it is possible to construct a map $G_{p,T}(f)$ and a map $G_{p,N}(f)$ of X into suitable Grassmannians in an analogous way to the first order case, which will be a *pth order Gauss tangent map* and a *pth order Gauss normal map of f* respectively. If $N\geqq\nu(n,p)+n+1$, $G_{p,T}(f)$ and $G_{p,N}(f)$ are the classifying maps for $T_p(X)$ and $N_{p,f}(X)$ respectively. If f_0 and f_1 are any two pth order nondegenerate immersions of X in $R^N(N\geqq\nu(n,p)+n+1)$ the homotopy between $G_{p,T}(f_0)$ and $G_{p,T}(f_1)$ and the homotopy between $G_{p,N}(f_0)$ and $G_{p,N}(f_1)$ given by the bundle classification theorem can be geometrically realized by the maps $G_{p,T}(f_t)$ and $G_{f,N}(f_t)$ given by a pth order regular homotopy.

5. **The pth fundamental form.** Let us return to the situation where Y and its connections are arbitrary.

DEFINITION. Let $f: X\to Y$ be a pth order nondegenerate immersion. Assume that $N>\nu(n,p)$. Let us denote by $f^{-1}(D_p)$ the "pull back" of the splitting D_p of

$$0\to T_p(Y)\to T_{p+1}(Y)\to 0^{p+1}T_1(Y)\to 0$$

to a splitting of

$$0\to f^{-1}T_p(Y)\to f^{-1}T_{p+1}(Y)\to 0^{p+1}f^{-1}T_1(Y)\to 0.$$

Let $\pi_{p,f}: f^{-1}T_1(Y)\to N_{p,f,Y}(X)$ be the canonical projection onto the cokernel of $\nabla_p T_p(f)!$. Consider the vector bundle homomorphism over X defined by $\pi_{p,f}f^{-1}(D_1)\cdots f^{-1}(D_p)T_{p+1}(f)!: T_{p+1}(X)\to N_{p,f,Y}(X)$. This homomorphism vanishes on $\operatorname{Im}(T_p(X)\to T_{p+1}(X))$. Hence there is induced a unique vector bundle homomorphism over X, $\nu_{p,f}: 0^{p+1}T_1(X)\to N_{p,f,Y}(X)$. $\nu_{p,f}$ is called the $(p+1)$*st fundamental form of the pth order immersion f.*

PROPOSITION. *Let* $f: X\to Y$ *be a pth order nondegenerate immersion. Let* $N>\nu(n,p)$.

(a) *If* $k<p$ *then* $\nu_{p,f}$ *is a monomorphism of vector bundles over* X.

(b) *If* $k\leqq p$ *then the following short exact sequences are satisfied*

$$0\to \nu_{k-1,f}(0^kT_1(X))\to N_{k-1,f,Y}(X)\to N_{k,f,Y}(X)\to 0.$$

COROLLARY. *If* $k\leqq p$ *then*

$$N_{1,f,Y}(X)\cong \nu_{1,f}(0^2T_1(X))\oplus\cdots\oplus \nu_{k-1,f}(0^kT_1(X))\oplus N_{k,f,Y}(X).$$

6. **Counterexamples.** In this section always assume that the target space Y equals R^N. However the connections on Y can be arbitrary. Standard characteristic class arguments [**3**; **4**; **5**] give us the following results.

THEOREM 6.1. (a) $P_2(R)$ *cannot be pth order nondegenerately immersed in* $R^{\nu(2,p)+1}$, *if* $p=8s+5$ *or if* $p=8s+3$, *for* s *any non-negative integer.*

(b) $P_2(C)$ *cannot be second order nondegenerately immersed in* $R^{\nu(4,2)+3}$.

(c) $P_2(R)$ *cannot be pth order nondegenerately immersed in* $R^{\nu(2,p)-1}$, *if* $p=8s+1$, *or if* $p=8s+3$, *for* s *any non-negative integer.*

REMARK. These examples show that Theorem 3.1 is in some sense optimal.

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