SIMPLY INVARIANT SUBSPACES

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Let $L^1$, $L^2$ denote respectively the spaces of summable and square summable functions on the circle group and $H^1$, $H^2$ their subspaces consisting of those functions whose Fourier coefficients vanish for negative indices. A closed subspace $M$ of $L^1$ or $L^2$ is "invariant" if

$$\chi M \subset M$$

and "simply invariant" if the above inclusion is strict, where $\chi$ is the character

$$\chi(x) = e^{ix}.$$

The structure of simply invariant subspaces is known, namely, they are precisely the subspaces of the form $qH^1$ or $qH^2$ (respectively) where $q$ is a measurable function of modulus 1 a.e. Beurling [1] first proved this for subspaces $M \subset H^2$; for $M \subset H^1$, this is due to de Leeuw-Rudin [5]; for $M \subset L^2$, due to Helson-Lowdenslager [3] and for $M \subset L^1$, due to Forelli [2]. In [3] Helson-Lowdenslager also gave a simple proof of the $H^2$ case, free of function theoretic considerations. Using their arguments Hoffman [4] extended this result to simply invariant subspaces of $H^2(dm)$ defined over logmodular algebras. In this paper we prove this result for simply invariant subspaces of $L^2(dm)$ and $L^1(dm)$ over logmodular algebras; the results of the previous authors follow as a corollary. The proofs of the previous authors

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do not extend to this general case as they depend on facts which either
have no analogues or are not true for the logmodular algebras; when
specialised to their contexts, our proof turns out to be even simpler.
Our proof for the case of $L^2(dm)$ was inspired by that of Helson-
Lowdenslager for the $H^2$ case and is in the same spirit as theirs.

Let $X$ be a compact Hausdorff space and $A$ a subalgebra of the
algebra $C(X)$ of complex continuous functions on $X$ with the uniform
norm.

$A$ is logmodular if

i. $A$ is uniformly closed,

ii. $A$ contains the constant functions,

iii. $A$ separates the points of $X$ and

iv. the set of functions $\log |f|$ where $f, 1/f \in A$, is uniformly dense
in the algebra of real continuous functions on $X$.

Let $m$ be a probability Baire measure on $X$ which is "multiplica-
tive" on $A$, meaning

$$\int fg \, dm = \int f \, dm \int g \, dm$$

for all $f, g \in A$ (such measures always exist), and let $H^1(dm), H^2(dm)$
denote the closures of $A$ in $L^1(dm), L^2(dm)$ respectively. The invariant
subspaces $M$ are now closed subspaces of $L^1(dm), L^2(dm)$, which are
invariant under multiplication by functions in $A$ or equivalently by
functions in $A_0$, where

$$A_0 = \{ f | f \in A, \int f \, dm = 0 \}$$

and the simply invariant $M$'s are those for which the inclusion
$A_0M \subset M$ is strict.\(^a\)

In the case considered earlier, $X$ was the unit circle, $A_0$ was the
uniform closure of the algebra generated by $x$ in $C(X)$ and $m$ the
normalised Lebesgue measure. We have

\textbf{Theorem.}

1. The simply invariant subspaces of $L^2(dm)$ are precisely the sub-
   spaces of the form $qH^2(dm)$ where $q \in L^2(dm)$ and $|q| = 1$ a.e. $(dm)$.

2. The simply invariant subspaces of $L^1(dm)$ are precisely the sub-
   spaces of the form $qH^1(dm)$ where $q \in L^1(dm)$ and $|q| = 1$ a.e. $(dm)$.

\(^a\) $A_0M$ should be replaced by its closure in $L^2(dm)$ respectively $L^1(dm)$, which
necessitates changes in the proof.

\(^b\) The details of the proof of the $L^1$ theorem and its function theoretic consequences
will be published separately.
PROOF. It is obvious that subspaces of the form \( qH^2(dm) \), \( qH^1(dm) \) are invariant; they are simply invariant because for instance, \( q \subseteq qH^2(dm) \), \( qH^1(dm) \) while \( q \not\subseteq qA_0H^2(dm) \), \( qA_0H^1(dm) \). To prove the converse:

1. We need the following facts about logmodular algebras [4, pp. 284, 293]:

   (a) \( A+\overline{A} \) is dense in \( L^2(dm) \) where the bar denotes complex conjugation,

   (b) if \( \mu \) is any positive Baire measure on \( X \) such that \( \int f d\mu = 0 \) for all \( f \in A_0 \) then \( d\mu = c dm \) for some constant \( c \).

   Now let \( M \subseteq L^2(dm) \) be simply invariant and let \( q \in M \cap A_0M \), \( q \neq 0 \). Then \( q \perp A_0 q \), so \( \int |q|^2 dm = 0 \) for all \( f \in A_0 \) and by (b), \( |q|^2 = c \) a.e. By modifying \( q \) we may assume that \( |q| = 1 \) a.e.

   Clearly \( qH^2(dm) \subseteq M \), because of invariance of \( M \). Let \( g \in M \cap qH^2(dm) \). Then \( g \perp q \), so \( g \perp A \). Also \( A_0 g \subseteq A_0 M \), so \( g \perp A_0 g \) so that \( g \perp A + \overline{A} \). Thus \( g \perp A + \overline{A} \), hence \( g = 0 \) a.e. by (a) and since \( |q| = 1 \) a.e., \( g = 0 \). Thus \( M = qH^2(dm) \).

2. We use (1) to prove (2). Let \( N \subseteq L^1(dm) \) be simply invariant and let \( M = N \cap L^2(dm) \). \( M \) is clearly an invariant subspace of \( L^2(dm) \). We shall show that it is actually simply invariant. Let \( f \in N \). We can find \( f_1, f_2 \in L^2(dm) \) such that \( f = f_1 f_2 \); we may also assume that one of them, say, \( f_2 \) is nonzero a.e. Then \( f_2H^2(dm) \) is a simply invariant subspace of \( L^2(dm) \) and is by (1) of the form \( q_2H^2(dm) \), \( |q_2| = 1 \) a.e. Now

   \[
   f_1 q_2 \subseteq f_1 q_2 H^2(dm) = f_1 f_2 H^2(dm) = f H^2(dm) \subseteq N.
   \]

   Also \( f_1 q_2 \in L^2(dm) \). Hence \( f_1 q_2 \subseteq M \). Suppose \( M = A_0 M \). Then \( f_1 q_2 \in A_0 M \). Let

   \[
   f_1 q_2 = f_0 g, \quad f_0 \in A_0, \quad g \in M \subseteq N
   \]

   and

   \[
   f = g h, \quad h \in H^2(dm).
   \]

   Then

   \[
   f = f_1 f_2 = f_1 q_2 h = f_0 g h \subseteq A_0 NH^2(dm) \subseteq A_0 N
   \]

   and it follows that \( N = A_0 N \). Hence if \( N \) is simply invariant, so is \( M \).

   Let then \( M = qH^2(dm) \) by (1). We shall show that \( N = qH^1(dm) \). Clearly \( qH^2(dm) \subseteq N \). Let \( f \in N \) and \( f_1, f_2, q_2, h \) be as above. Then \( f_1 q_2 \subseteq M = qH^2(dm) \). Let \( f_1 q_2 = g h', h' \in H^2(dm) \). Then

   \[
   f = f_1 f_2 = f_1 q_2 h = q h' h \subseteq qH^1(dm)
   \]

   as \( h', h \in H^2(dm) \). It follows that \( N = qH^1(dm) \).
We may remark that if $M \subseteq \mathcal{H}^1(dm)$ is invariant and we assume with Hoffman [4, p. 293] that $\int g dm \neq 0$ for at least one $g \in M$ then $M$ is certainly simply invariant and Hoffman's result follows. But this latter condition is not necessary for simple invariance as the example of $a^k \mathcal{H}^1$, $k \geq 1$ shows.

REFERENCES


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