A BIORTHOGONAL SYSTEM WHICH IS NOT A TOEPLITZ BASIS

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We present a Banach space X with a biorthogonal system $\{b_n, f_n\}$, $b_n \in X$, $f_n \in X^*$, $f_i(b_j) = \delta_{ij}$, in which $\{b_n\}$ is fundamental, i.e., X is its linear closure, but $\{b_n\}$ is not a Toeplitz basis. We call $\{b_n\}$ a Toeplitz basis if there exists a regular matrix A such that for each $x \in X$, $\sum f_n(x) b_n$ is A-summable to x. It will be clear that $\{b_n\}$ is not a Toeplitz basis even if the method of summability is considerably more general than a matrix.

An example in which X is a non-Banach F space is given in [**4**, Theorem 13]. Toeplitz bases were introduced in [**1**], [**2**].

A series $\sum u_n$ is said to be weakly Cauchy if for every continuous linear functional f, $\sum f(u_n)$ is convergent.

LEMMA 1. *Let X be a linear topological space with a biorthogonal system $\{b_n, f_n\}$. Suppose that there exists $x_0 \in X$ such that $\sum f_n(x_0) b_n$ is weakly Cauchy. Then if $\{b_n\}$ is a Toeplitz basis, $\sum f_n(x_0) b_n$ must converge weakly to x_0.*

For any $f \in X^*$, $\sum f_n(x_0) f(b_n)$ is convergent. It must converge to $f(x_0)$ since, for some regular matrix A, it is A-summable to $f(x_0)$.

The full force of the hypothesis is not used. It can be considerably weakened.

From Lemma 1 it follows that the promised example will be fulfilled by an example of a Banach space X of sequences $x = \{x_n\}$ in which, for each n, $f_n(x) = x_n$ defines $f_n \in X^*$; in which $\{\delta^n\}$ is fundamental, where $\delta^1 = (1, 0, 0, \cdots)$, $\delta^2 = (0, 1, 0, \cdots)$, $\cdots$; in which $\sum \delta^n$ is weakly Cauchy; but in which $\sum \delta^n$ does not converge weakly to 1.

Let $B = (b_{nk})$ be a matrix which is a triangle, i.e., $b_{nn} \neq 0$, $b_{nk} = 0$ for $k > n$; which is coregular, i.e., $\chi(B) \equiv \lim_B 1 - \sum \lim_B \delta^k \neq 0$ where $\lim_B x = \lim_n \sum_k b_{nk} x_k$; for which $\{\delta^n\}$ is fundamental in $c_B = \{x : x \text{ is } B \text{ summable}\}$, $\|x\| = \sup_n \left| \sum_k b_{nk} x_k \right|$; but which has the property that no regular matrix D has $c_D = c_B$. (See [**3**, p. 657] for an example.)

Then $\sum \delta^n$ is weakly Cauchy since c_B has a weaker topology than that of c_0, the space of null sequences with $\|x\| = \sup |x_n|$, and, in the latter space $\sum \delta^n$ is weakly Cauchy.

If $\{\delta^n\}$ were a Toeplitz basis, by Lemma 1, we should have $\sum \delta^n$

converging weakly to 1. Since $\lim_B \in c_B^*$ this would yield $\chi(B)=0$, contradicting the fact that B is coregular.

Since c_B is congruent with c, the space of convergent sequences with $\|x\| = \sup |x_n|$ we have proved the following result.

THEOREM 1. *The space c has a biorthogonal system $\{b_n, f_n\}$ in which $\{b_n\}$ is fundamental but is not a Toeplitz basis.*

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