# FREDHOLM EIGENVALUES AND QUASICONFORMAL MAPPING[1]

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Let $\tilde{D}$ be a region of connectivity $n$ in the $z$-plane which contains the point at infinity. Suppose that its boundary consists of $n$ Jordan curves $C_1, C_2, \cdots, C_n$, each of which is given parametrically in terms of its arc length by functions having continuous second derivatives which satisfy a Hölder condition of order $\alpha$ $(0 < \alpha \leqq 1)$. Each $C_j$ $(j=1, \cdots, n)$ forms the boundary of a simply connected bounded region $D_j$, and we shall write $D=D_1 \cup \cdots \cup D_n$ and $C=C_1 \cup \cdots \cup C_n$.

Let $\lambda$ denote the smallest eigenvalue satisfying $\lambda > 1$ of the integral equation

$$f(z) = \frac{\lambda}{\pi} \int_C f(t) \frac{\partial}{\partial n_t} \log \frac{1}{|z-t|} ds_t, \qquad t \in C,$$

where $s_t$ denotes the arc length parameter on $C$ oriented positively with respect to $\tilde{D}$ and $\partial/\partial n_t$ denotes differentiation in the direction of the normal to $C$ pointing into $\tilde{D}$. We shall refer to $\lambda$ as the Fredholm eigenvalue of $C$.

We next suppose that $\zeta(z)$ is a quasiconformal homeomorphism of the $z$-sphere onto the $\zeta$-sphere $(\infty \to \infty)$ whose generalized derivatives are denoted by

$$p = \frac{\partial \zeta}{\partial z} = \frac{1}{2}\left(\frac{\partial \zeta}{\partial x} - i \frac{\partial \zeta}{\partial y}\right) \quad \text{and} \quad q = \frac{\partial \zeta}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial \zeta}{\partial x} + i \frac{\partial \zeta}{\partial y}\right).$$

We assume that $\zeta(z)$ is $K$-quasiconformal in $D$ and $M$-quasiconformal in $\tilde{D}$; i.e., $\|q\|_\infty \leqq k\|p\|_\infty$ $(k<1)$ in $D$ and $\|q\|_\infty \leqq m\|p\|_\infty$ $(m<1)$ in $\tilde{D}$, where $K=(1+k)/(1-k)$ and $M=(1+m)/(1-m)$. The image of each $C_j$ is a curve $C_j^*$ and the curve system $C=C_1^* \cup \cdots \cup C_n^*$ has Fredholm eigenvalue $\lambda^*$. The following inequality holds:

THEOREM.

$$\frac{\lambda^*+1}{\lambda^*-1} \leqq KM \frac{\lambda+1}{\lambda-1}. \tag{1}$$

If $\lambda$ is known and the $K$ and $M$ for the mapping $\zeta$ are known, then

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(1) yields a lower bound for $\lambda^*$, estimates of which are important in many problems in potential theory and conformal mapping.

If $C$ consists of only one circle, then $\lambda = \infty$ and (1) becomes

$$(\lambda^* + 1)/(\lambda^* - 1) \leqq KM. \tag{2}$$

In this case, the images of $D$ and $\tilde{D}$ are both simply connected and if $\rho(z)$ denotes the reflection mapping in the circle $C$ (a sense-reversing mapping with maximal dilatation one), then the mapping $\zeta \circ \rho \circ \zeta^{-1}$ is a $KM$-quasiconformal reflection [2] in the curve $C^*$. In particular, if $\zeta(z)$ is conformal in $\tilde{D}$ (i.e., $M=1$), then we obtain the estimate $\lambda \geqq 1/k$, which was earlier obtained by Ahlfors [1]. This shows that (1) may be viewed as a generalization of the Ahlfors result to multiply connected regions.

The inequality (1) is obtained by studying the effect of the quasiconformal mappings on the Dirichlet integrals appearing in the extremal characterization of the eigenvalue $\lambda$.

## REFERENCES

1. L. V. Ahlfors, *Remarks on the Neumann-Poincaré integral equation*, Pacific J. Math. 2 (1952), 271–280.

2. ———, *Quasiconformal reflections*, Acta Math. 109 (1963), 291–301.

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