NOTE ON LINEAR DIFFERENCE EQUATIONS

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Communicated by Wolfgang Wasow, September 25, 1963

Particular solutions of nonlinear differential equations have been used successfully to achieve analytic simplification of systems of linear differential equations [7; 8]. In this note we will show that similar results are possible for systems of linear difference equations. To the author's knowledge, this is the first time this technique has been employed for difference equations.

We are concerned with the system of linear difference equations

$$y(x+1) = x^{\mu}A(x)y(x), \tag{1}$$

where y is a vector with n components, μ is an integer, and $A(x)$ is an n by n matrix with elements analytic in a neighborhood of $x=\infty$:

$$A(x) = \sum_{s=0}^{\infty} A_s x^{-s}, \qquad |x| > \rho, \ A_0 \neq 0.$$

The most effective manner for determining the solutions formally[3] is to reduce the difference equation (1) into k systems of the same type and of lower order by a formal transformation[4] of the form

$$y(x) = T(x)z(x) \tag{2}$$

where

$$T(x) = \sum_{s=0}^{\infty} T_s x^{-s} \text{ (formally), det. } T_0 \neq 0.$$

More precisely speaking, let the resulting equation be

$$z(x+1) = C(x)z(x)$$

where $T(x)$ has been constructed so that $C(x)$ has the block diagonal form

$$C(x) = (C_1(x), C_2(x), \cdots, C_k(x)),$$

with

$$C_i(x) = x^{\mu_i} \sum_{j=0}^{\infty} C_{ij}x^{-j}, \qquad C_{i0} = \lambda_i I_i + N_i.$$

[1] Supported in part by the National Science Foundation under Grant G-18918.

[2] Supported in part by a contract of the Office of Naval Research, Nonr-3776(00).

[3] For the direct construction of formal solutions see [1; 2; 3].

[4] For the construction of the formal transformation and the resulting canonical form see [9; 10].

Here μ_i are integers, λ_i are constants, I_i are unit-matrices, N_i are nilpotent matrices, and $\mu_i=\mu_j$ implies $\lambda_i\neq\lambda_j$.

The formal transformation $T(x)$ will in general be divergent, but in appropriate sectors of the x-plane desirable analytic properties are available which in turn will also be available for $C(x)$, since

$$C(x) = x^{\mu}T^{-1}(x+1)A(x)T(x).$$

In this note we establish the following result.

THEOREM. *Let the elements of the n by n matrix $A(x)$ be analytic for $|x|>\rho$, $A(x)=\sum_{s=0}^{\infty} A_s x^{-s}$, $A_0=A(\infty)\neq 0$ and let the matrix A_0 have the block diagonal form*

$$A_0 = \operatorname{diag}(A_1^0, \cdots, A_p^0)$$

where the eigenvalues λ_{ij} of A_i^0 satisfy the conditions

$$|\lambda_{ij}| = |\lambda_{il}|; \qquad |\lambda_{ij}| \neq |\lambda_{hl}|; \qquad i \neq h.$$

Then there exists a matrix $T(x)$ with elements analytic for $\operatorname{Im} x\geqq R>0$ *and* $\operatorname{Im} x\leqq -R<0$ *if R is sufficiently large for which*

$$T^{-1}(x+1)A(x)T(x) = B(x) = \operatorname{diag}(B^1(x), \cdots, B^p(x)).$$

Further $T(x)$ has the asymptotic representation

$$T(x) \cong I + \sum_{s=1}^{\infty} T_s x^{-s}$$

for the regions $\operatorname{Im} x\geqq R>0$ *and* $\operatorname{Im} x\leqq -R<0$, *hence*

$$B(x) \cong \sum_{s=0}^{\infty} B_s x^{-s}, \qquad B_0 = A_0.$$

If we assume further that A_0 is a diagonal matrix (or reducible in the sense of [5]), the results of the authors [6] give the following corollary.

COROLLARY. *If A_0 is diagonal, the block diagonalization of the theorem may be refined so that different blocks correspond to distinct eigenvalues of the matrix A_0 with $B(x)$ having asymptotic representations in the sectors*

$$|\operatorname{Im} x| \geqq R, \qquad |\arg x - k\pi| \leqq \frac{\pi}{2} - \epsilon,\ \epsilon > 0, \quad k = 0, 1.$$

PROOF OF THE THEOREM. We assume without loss of generality that A_j^0 is in Jordan canonical form

$$(3)\qquad \overset{0}{A}_j = \begin{bmatrix} \lambda_{j1}\delta_{j1} & & \\ & \ddots & \\ & & \delta_{jm_j-1} \\ & & \lambda_{jm_j} \end{bmatrix}, \qquad j = 1, \cdots, p,$$

with δ_{jh} arbitrary small.

Let $T(x) = I + Q(x)$, $A(x) = A_0 + \hat{A}(x)$ and $B(x) = A_0 + F(x)$. Then $T^{-1}(x+1)A(x)T(x) = B(x)$ becomes

$$(4)\qquad \begin{aligned} \Delta Q(x)A_0 = {} & A_0Q(x) - Q(x)A_0 + A(x)Q(x) - Q(x)F(x) + A(x) \\ & - F(x) - \Delta QF(x) \end{aligned}$$

where $\Delta Q(x) = Q(x+1) - Q(x)$. Put

$$A = \begin{bmatrix} \hat{A}_{11} & \cdots & \hat{A}_{1p} \\ \vdots & & \vdots \\ \hat{A}_{p1} & \cdots & \hat{A}_{pp} \end{bmatrix}, \qquad F = \begin{bmatrix} F_1 & & 0 \\ & \ddots & \\ 0 & & F_p \end{bmatrix}$$

$$Q = \begin{bmatrix} 0 & Q_{12} & Q_{13} & \cdots & Q_{1p} \\ Q_{21} & 0 & Q_{23} & \cdots & Q_{2p} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ Q_{p1} & & \cdots & Q_{pp-1} & 0 \end{bmatrix}.$$

Then $F_j = \hat{A}_{jj} + \sum_{s \neq j} \hat{A}_{js}Q_{sj}$ and the equation for determining Q becomes

$$(5)\qquad \begin{aligned} \Delta Q_{js}\overset{0}{A}_s = {} & \overset{0}{A}_jQ_{js} - Q_{js}\overset{0}{A}_s + \sum_{h \neq s} A_{jh}Q_{hs} - \{\Delta Q_{js} + Q_{js}\} \\ & \cdot \Big\{A_{ss} + \sum_{h \neq s} A_{sh}Q_{hs}\Big\} + A_{js}. \end{aligned}$$

Assume, if necessary (i.e., some zero eigenvalues), that A_1^0 is singular. If Q is determined in this manner, then T and F are also determined. Equation (5) is a system of nonlinear difference equations. Hence we are led to the following problem:

Let y and z be α and β dimensional vectors respectively and consider the following nonlinear difference equation

$$(6)\qquad \begin{aligned} \Delta y &= f(x, y, z, \Delta y), \\ C_0\Delta z &= g(x, y, z, \Delta z), \end{aligned}$$

where

$$f = f_0(x) + P_0y + \hat{f}(x, y, z, \Delta y), \qquad g = g_0(x) + Q_0z + \hat{g}(x, y, z, \Delta z);$$

P_0 and Q_0 are nonsingular constant matrices, C_0 is a constant singular matrix whose eigenvalues are all zero and f_0, g_0, $\hat{f}$, and $\hat{g}$ are of the form $O(x^{-1})$; construct a solution of (6) which has an asymptotic representation in powers of x^{-1} in an appropriate sector.

We obtain such a solution by showing the existence of a fixed point of a mapping in a certain function space.

The inequality $|\lambda_{jh}| \neq |\lambda_{lm}|$ if $j \neq l$ allows us to write

$$P_0 + I = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \tag{7}$$

where the eigenvalues of P_1 and P_2 have absolute value less than and greater than one respectively.

Let

$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \qquad f_0 = \begin{bmatrix} f_{01} \\ f_{02} \end{bmatrix} \quad \text{and} \quad \hat{f} = \begin{bmatrix} \hat{f}_1 \\ \hat{f}_2 \end{bmatrix}$$

correspond to the partitioning (7). Then the existence of $T(x)$ is equivalent to the existence of a fixed point of the mapping

$$\begin{bmatrix} \phi_1 \\ \phi_2 \\ \psi \end{bmatrix} \to \begin{bmatrix} y_1 \\ y_2 \\ z \end{bmatrix}$$

which is defined as follows:

$$\begin{aligned} y_1(x) &= f_{01}(x-1) + P_1\phi_1(x-1) \\ &\qquad + \hat{f}_1(x-1, \phi(x-1), \psi(x-1), \Delta\phi(x-1)), \\ y_2(x) &= P_2^{-1}\{-f_{00}(x) + \phi_2(x+1) - \hat{f}_2(x, \phi(x), \psi(x), \Delta\phi(x))\}, \\ z(x) &= -Q_0^{-1}\{g_0(x) - C_0\Delta\psi + \hat{g}(x, \phi(x), \psi(x), \Delta\psi(x))\}. \end{aligned}$$

Let $\mathfrak{F}$ be the set of all vector-valued functions

$$\Phi = \begin{bmatrix} \phi_1(x) \\ \phi_2(x) \\ \psi(x) \end{bmatrix}$$

whose components are holomorphic for $\operatorname{Im} x > R > 0$ (or $\operatorname{Im} x < -R < 0$) and satisfy the inequality

$$\|\Phi(x)\| \leqq M^{5}$$

[5] The norm of the vector y with components $y_1 \cdots y_n$ is defined by $\|y\| = \sum_{i=1}^{n} |y_i|$. If A is an n by n matrix the norm of A is defined by $\|A\| = \sup\{\|Ay\|; \|y\| = 1\}$.

where M is an arbitrary but fixed constant not depending on Φ. $\mathfrak{F}$ is closed, compact, and convex with respect to the topology of uniform convergence on each compact subset of the indicated region. Since the mapping is continuous we only need to show that $\mathfrak{F}$ is mapped into $\mathfrak{F}$.

If the δ_{jh} in (3) are sufficiently small we have

$$\|P_1\| < 1, \qquad \|P_2^{-1}\| < 1, \quad \text{and} \quad \|Q_0^{-1}C_0\| < 1.$$

Utilizing this fact, we can choose R so that $\mathfrak{F}$ is mapped into $\mathfrak{F}$. Thus we can establish the existence of a bounded solution of (6). The asymptotic properties of this solution can be proved in a manner analogous to the proof of case (5) in [6] and will be omitted.

We note that the results of the theorem are valid if we replace the condition that the elements of $A(x)$ are analytic for $|x| > \rho$ by the condition that $A(x)$ is analytic for $|\operatorname{Im} x| > \rho$ and has an asymptotic expansion $A(x) \cong \sum_{s=0}^{\infty} A_s x^{-s}$.

Further results pertaining to properties of particular solutions of nonlinear difference equations will simplify and extend results for linear systems of difference equations. For example, the Borel summability of solutions of linear systems of difference equations may be studied conveniently if the Borel summability of the transformation $T(x)$ can be established [4; 5].

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