SCATTERING THEORY

BY PETER D. LAX[1] AND RALPH S. PHILLIPS[2]

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1. Let H be a Hilbert space, $U(t)$ a group of unitary operators. A closed subspace D_+ of H will be called *outgoing* if it has the following properties:

(i) $U(t)D_+ \subset D_+$ for t positive.
(ii) $\bigcap_{t>0} U(t)D_+ = \{0\}$.
(iii) $\bigcup_{t<0} U(t)D_+$ dense in H.

A prototype of the above situation is when H is $L_2(-\infty, \infty; N)$, i.e., the space of square integrable functions on the whole real axis whose values lie in some accessory Hilbert space N, $U(t)$ is translation by t, and D_+ is $L_2(0, \infty; N)$.

THEOREM 1.[3] *If D_+ is outgoing for the group $U(t)$, then H can be represented isometrically as $L_2(-\infty, \infty; N)$ so that $U(t)$ is translation and D_+ is the space of functions with support on the positive reals. This representation is unique up to isomorphisms of N.*

We shall call this representation an *outgoing translation representation* of the group.

Taking the Fourier transform we obtain an *outgoing spectral representation* of the group $U(t)$, where elements of D_+ are represented as functions in $A_+(N)$, that is the Fourier transform of $L(0, \infty; N)$. According to the Paley-Wiener theorem $A_+(N)$ consists of boundary values of functions with values in N, analytic in the upper half-plane whose square integrals along lines $\operatorname{Im} z = \text{const}$ are uniformly bounded.

An incoming subspace D_- is defined similarly and an analogous representation theorem holds, D_- being represented by functions with support on the negative axis, that is, by $L_2(-\infty, 0; N_-)$. N_- and N are unitarily equivalent and will henceforth be identified. In the application to the wave equation there is a natural identification of N and N_-.

Let D_+ and D_- be outgoing and incoming subspaces respectively for the same unitary group, and suppose that D_+ and D_- are *orthogonal*. To each function $f \in H$ there are associated two functions k_- and k_+, the respective incoming and outgoing translation representa-

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[3] We were informed by Professor Sinai that he has obtained and used a similar theorem.

tions of f. The mapping $k_- \to k_+$, denoted by S, is called the *scattering operator* and has the following properties:

(i) S is unitary.

(ii) S commutes with translation.

(iii) S maps $L_2(-\infty, 0, N)$ into $L_2(-\infty, 0, N)$.

Properties (i) and (ii) follow from the fact that S is defined in terms of two different unitary translation representations of the same group. To deduce property (iii), we note that every function in $L_2(-\infty, 0, N)$ of the incoming representation corresponds to an element f of D_-; since we have assumed that D_- is orthogonal to D_+ it follows that the function representing f in the outgoing representation will be orthogonal to $L_2(0, \infty, N)$, i.e., will belong to $L_2(-\infty, 0, N)$, as asserted in (iii).

We take now Fourier transforms and define the operator $\mathcal{S}$ as FSF^{-1}, F denoting the Fourier transformation. Properties (i)–(iii) for S translate into

(i)′ $\mathcal{S}$ is unitary.

(ii)′ $\mathcal{S}$ commutes with multiplication by scalar functions.

(iii)′ $\mathcal{S}$ maps $A(N)$ into $A(N)$

where $A(N)$ is the Fourier transform of $L_2(-\infty, 0, N)$ and thus consists of the boundary values of functions analytic in the lower half-plane.

According to a simple special case of a theorem of Segal and Fourès [13], an operator with properties (i)′, (ii)′ and (iii)′ is multiplication by an operator valued function $\mathcal{S}(z)$, mapping N into N, with the following properties:

THEOREM 2. (a) *$\mathcal{S}(z)$ is analytic in the lower half-plane.*

(b) *The norm of $\mathcal{S}(z)$ is not greater than one for every z.*

(c) *$\mathcal{S}(z)$ is unitary for z real.*

$\mathcal{S}(z)$ is the Heisenberg *scattering matrix*. Extending the terminology of Beurling [1], to the operator case $\mathcal{S}(z)$ is also an *inner factor*.[4]

2. Let D_+ and D_- be as before and denote by P_+ and P_- orthogonal projection onto the orthogonal complements of D_+ and D_- respectively. Consider the one-parameter family of operators $Z(t)$ defined as

$$Z(t) = P_+ U(t) P_-.$$

It follows easily that for positive t, $Z(t)$ annihilates both D_+ and D_-; consider $Z(t)$ for positive values of t and acting on $K = H \ominus D_+ \ominus D_-$.

THEOREM 3. *$Z(t)$ forms a semigroup over K.*

[4] See [1], [7] and [2] for the theory of inner factors.

This is very easy to prove directly from the postulated relations of D_+ and D_- to each other and $U(t)$. It also follows from the interpretation of $Z(t)$ in the, say, outgoing translation representation. For, since $H \ominus D_+$ is represented by $L_2(0, \infty, N)$ and D_- by $L_2(0, \infty, N)$, K is represented by

$$K \Leftrightarrow L_2(0, \infty, N) \ominus L_2(0, \infty, N),$$

and *the action of $Z(t)$ consists in shifting to the right followed by restriction to the negative real axis.*

A subspace of functions K which is mapped into itself under such an operation is called a *translation invariant space.* It is not surprising that K and $Z(t)$ can be so represented, since according to a simple generalization of a theorem of Beurling every contraction semigroup $Z(t)$ can be so represented, provided that for every u in K, $\|Z(t)u\|$ tends to zero as t tends to infinity.

In the outgoing spectral representation K is represented as

$$A(N) \ominus \mathcal{S}A(N);$$

$\mathcal{S}$ is called the *inner factor* associated with the translation invariant space K. Again this is no surprise since according to a generalization due to Lax, [6], [7] of a theorem of Beurling, see also Halmos [2], the orthogonal complement of the Fourier transform of every translation invariant space is of the form $\mathcal{S}A$, where $\mathcal{S}$ is an inner factor. The importance of this representation is that the associated inner factor contains almost complete information about the spectrum of $Z(t)$ over K:

THEOREM 4.[5] (a) *Let μ be a complex number with negative real part; μ belongs to the resolvent set of the infinitesimal generator B of $Z(t)$ if and only if the operator*

$$\mathcal{S}(i\bar{\mu})$$

is invertible.

(b) *Let λ be a complex number of absolute value less than one; λ belongs to the resolvent set of $Z(t)$ if and only if*

$$\mathcal{S}\left(\frac{i}{t}\bar{\mu}\right)$$

is invertible for all numbers μ for which $e^{\mu t}=\lambda$, and if the norms of the inverses are uniformly bounded for all such μ.

As corollary we obtain another proof of the well-known result of Phillips, see [11] or [3], that if μ belongs to the spectrum of B then

[5] See [10] for the scalar case.

$e^{\mu t}$ belongs to the spectrum of $Z(t)$, but in general not conversely.

THEOREM 5. *If for some value of* T, $\|Z(T)\| = a < 1$, *then* $\mathcal{S}(z)$ *can be continued analytically into the strip* $\operatorname{Im} z \leqq -\log a/T$.

Analogous expressions can be derived for the location of the spectrum of any function of $Z(t)$ and B; from this we deduce

THEOREM 5′. *If for some* T *and some* μ, $Z(T)(B-\mu)^{-1}$ *is completely continuous, then* $\mathcal{S}(z)$ *can be continued into the upper half-plane as a meromorphic function.*

The definition of the scattering matrix depends on the choice of a pair of orthogonal incoming and outgoing subspaces. Let us call two outgoing subspaces D_+ and D'_+ *equivalent* if for sufficiently large positive T, $U(-T)D_+$ contains D'_+ and $U(T)D_+$ is contained in D'_+. Following the derivation of Theorem 2 one can easily show that the outgoing spectral representations with respect to D_+ and D'_+ are related by multiplication by an operator valued function $\mathfrak{M}(z)$ which is entire analytic, of exponential growth, and unitary on the real axis. Such a function satisfies the relation

$$\mathfrak{M}^*(\bar{z}) = \mathfrak{M}^{-1}(z)$$

for z real, so by analytic continuation for all z; this shows that $\mathfrak{M}^{-1}$ exists for all z and is of exponential growth.

Suppose that D_+, D_- and D'_+, D'_- are two pairs of orthogonal incoming and outgoing subspaces which are equivalent. Then the associated scattering matrices are related by

$$\mathcal{S}' = \mathfrak{M}_+ \mathcal{S} \mathfrak{M}_-^{-1}.$$

Since the factors $\mathfrak{M}_+$, $\mathfrak{M}_-$ and their inverses are uniformly bounded in any strip, we conclude from Theorem 4 that the associated semigroups have the same spectrum.

Choose in particular D^a_+ as $U(a)D_+$, and D^a_- as $U(-a)D_-$, a positive. As is easily shown,

$$\mathcal{S}^a(z) = e^{2iaz}\mathcal{S}(z).$$

We denote the operator $Z(t)$ corresponding to $D^a_\pm$ by $Z_a(t)$.

THEOREM 6. *If* f^b *is an eigenvector of* $Z_b(t)$ *with eigenvalue* $e^{\mu t}$, *then for* $a < b$

$$f^a = P^a_+ f^b$$

is an eigenvector of $Z_a(t)$ *with the same eigenvalue.*

3. Let H_0 denote the Hilbert space of pairs of functions $f=[f_1, f_2]$ defined in R_n, normed by the energy norm:

$$\|f\|^2 = \int (|Df_1|^2 + |f_2|^2)\, dx.$$

Define $U_0(t)$ as the operator which relates the Cauchy data at time zero of solutions of the wave equation to their Cauchy data at time t. $U_0(t)$ forms a one-parameter group of unitary operators mapping H_0 onto H_0 (conservation of energy).

Consider a smooth, bounded, reflecting obstacle. Denote by H the subspace of H_0 consisting of pairs of functions which vanish inside the obstacle, and denote by $U(t)$ the operator which relates the initial data to data at time t of solutions of the wave equation defined outside of the obstacle and vanishing on it. $U(t)$ forms a one-parameter group of unitary operators mapping H onto H.

We shall call a solution of the wave equation defined for all values of x and t *outgoing* (*incoming*) if it vanishes inside the cone $|x| < t$ ($|x| < -t$). We denote by $D^0_\pm$ the data at time zero of outgoing (incoming) solutions.

D^0_+ and D^0_- are outgoing and incoming subspaces for the group $U_0(t)$ in the sense of §1; the first two properties are obviously satisfied and the third is an easy consequence of Huygens' principle. As shown in [9], D^0_+ and D^0_- are orthogonal for n odd; we give here a new proof based on an explicit form for the translation representation.

We start with the representation of functions in terms of their Radon transforms:

$$f(x) = \int_{|\omega|=1} h(x\cdot\omega, \omega)\, d\omega \tag{3.1}$$

where $h(s, \omega)$, the Radon transform of f, is a function of s and ω defined for all real s and all vectors ω on S_{n-1} which is even:

$$h(-s, -\omega) = h(s, \omega).$$

A Parseval relation holds:

$$\|f\|^2 = \|h\|_{-(n-1)/2}, \tag{3.2}$$

where we define

$$\|h\|^2_{-q} = \int |k(s, \omega)|^2 ds\, d\omega, \qquad \frac{\partial^q}{\partial s^q} k = h.$$

COROLLARY.

$$\|f\|_1 = \|h\|_{-(n-3)/2} \tag{3.2$'$}$$

Let h_1 and h_2 denote the Radon transforms of f_1 and f_2 respectively and define h as

$$h = h_1 - \int h_2.$$

It can be verified immediately that the function

$$u(x, t) = \int h(x\cdot\omega - t, \omega)d\omega \tag{3.3}$$

is a solution of the wave equation and that its initial data are f_1 and f_2. Furthermore, by (3.2) and (3.2)$'$,

$$\|f\| = \|h\|_{-(n-3)/2} \tag{3.4}$$

Let k be the $(n-3)/2$ fold integral of h; regarding k as a function of s whose values lie in the Hilbert space $N = L_2(S_{n-1})$, we conclude from (3.3), (3.4) that $f\to k$ is a translation representation for $U_0(t)$. We claim that for n odd this representation is both incoming and outgoing.

It follows from (3.3) that if $h(s)$ vanishes for negative (positive) values of s, then $u(x, t)$ vanishes in the forward (backward) cone $|x| < t$ ($|x| < -t$). Conversely:

THEOREM 7. *If $u(x, t)$ vanishes in the forward (backward) cone then h vanishes on the negative (positive) axis.*

SKETCH OF PROOF. If u vanishes in the forward cone, all its space derivatives vanish on the positive t axis:

$$0 = (D_x^j u)(0, t) = \int \omega^j h^{|j|}(-t, \omega)d\omega.$$

Multiply this by any smooth test function $\phi(t)$ whose support lies on the positive t-axis, integrate with respect to t and perform $|j|$ integrations by parts:

$$0 = \int \omega^j h(-t, \omega)\phi^{|j|}(t)d\omega dt. \tag{3.5}$$

From (3.5) and the fact that h has finite $(3-n)/2$ norm it follows by an approximation procedure that for every smooth test function $\chi(t)$ with compact support on the positive t axis and every multi index j

$$\int\int h(-t,\,\omega)\omega^j\chi(t)d\omega dt = 0.$$

But this implies that $h(s)$ vanishes for s negative; then so does k for n odd.

REMARK. In the proof we only used the fact that $u(x,\,t)$ has a zero of infinite order on the positive t-axis; thus we have shown that this condition implies that u vanishes in the forward cone—a new proof for a special case of a theorem of Fritz John.

COROLLARY. *For n odd, D_+ and D_- are orthogonal.*

For the group $U_0(t)$ we have found a representation with D_+^0 and D_-^0 as outgoing and incoming subspaces. The associated scattering operator is the identity. We turn now to the group $U(t)$ and take for D_+ and D_- the initial data of solutions which vanish in $|x| < \rho + t$ for $t > 0$ and $|x| < \rho - t$ for $t < 0$ respectively;[6] we claim that these subspaces are outgoing and incoming respectively for $U(t)$: Properties (i) and (ii) are immediate while property (iii) is proved in [8]. D_+ and D_- are orthogonal since they are subspaces of D_+^0 and D_-^0. *Thus there exists an associated scattering matrix.* Conversely, we can prove

THEOREM 8. *The scattering matrix uniquely determines the scattering obstacle.*

In [9], Cathleen Morawetz and the authors have shown that for star-shaped obstacles $\|Z(t)\|$ is less than one for t large enough. By Theorem 5 it follows that the associated scattering matrix can be continued analytically into a strip $0 \leqq \operatorname{Im} z \leqq \tau$. For any obstacle we have this result:

THEOREM 9. *For* $\operatorname{Re}\lambda$ *positive* $Z(2\rho)(B-\lambda)^{-1}$ *is completely continuous.*

By Theorem 5′ this implies that $\mathcal{S}(z)$ can be continued into the upper half-plane as a meromorphic function. This implies that the zeros of $\mathcal{S}(z)$ in the lower half-plane are discrete; furthermore, for each z, $\mathcal{S}(z)$ has a closed range whose codimension is finite and equal to the dimension of the nullspace of $\mathcal{S}(z)$.

SKETCH OF PROOF.

LEMMA 1. *The operator*

$$M = U(2\rho) - U_0(2\rho)$$

[6] ρ is chosen so large that the sphere $|x| < \rho$ contains the obstacle.

annihilates all f in H which vanish in $|x| < 3\rho$.

LEMMA 2. *$U_0(2\rho)$ maps the complement of D_- into D_+.*

Lemma 2 implies that $P_+U_0(2\rho)P_- = 0$, whence for $t \geqq 2\rho$

$$Z(t) = P_+U(t)P_- = P_+MU(t - 2\rho)P_-. \tag{3.6}$$

So

$$\begin{aligned} Z(2\rho)(B - \lambda)^{-1} &= e^{2\lambda\rho}\int_{2\rho}^{\infty} Z(t)e^{-\lambda t}dt \\ &= e^{2\lambda\rho}P_+M\int_{2\rho}^{\infty} U(t - 2\rho)e^{-\lambda t}dtP_- = P_+M(A - \lambda)^{-1}P_-, \end{aligned} \tag{3.7}$$

where A denotes the infinitesimal generator of $U(t)$. It is easy to show that $(A - \lambda)^{-1}$ raises by one the degree of differentiability; since by Lemma 1 the value of Mf does not depend on the values of f outside the sphere $|x| \leqq 3\rho$, it follows by Rellich's compactness criterion that $M(A - \lambda)^{-1}$ is a completely continuous operator.

For the pure initial value problem for hyperbolic equations with variable coefficients it is known that the sharp propagation of signals is along characteristic rays. This *generalized Huygens principle* can be reformulated as follows:

Let G_1 and G_2 be two closed sets in R_n with the property that no characteristic ray starting at time zero in G_1 passes through G_2 at time t. Let P_1 and P_2 be operators such that the range of P_1 consists of functions which vanish outside of G_1, while P_2 annihilates all functions whose support lies outside G_2. Then

$$P_2U_0(t)P_1,$$

is completely continuous.

We believe that this principle also holds for the mixed problem as well (for general hyperbolic equations with variable coefficients), provided that rays are interpreted as reflected rays. A demonstration of this for the interior problem for convex domains has been given by Povsner and Suharevskiĭ [12].

We say that an obstacle has property L if there exists a number l such that any ray starting in the sphere $|x| \leqq 3\rho$ leaves the sphere $|x| \leqq 3\rho$ after time l.

REMARK. Star-shaped obstacles have property L.

Assuming the generalized Huygens principle to hold we assert:

THEOREM 10. *$Z(t)$ is eventually compact if and only if the obstacle has property L.*

PROOF. The identity (see [9])

$$Z(t + 4\rho) = P_+ M U(t) M P_- \tag{3.6$'$}$$

follows similarly as (3.6). Take both G_1 and G_2 as the sphere $|x| \leqq 3\rho$; the operator MP_- has the property required of P_1 while M has the property required of P_2. So by the generalized Huygens principle $Z(l+4\rho)$ is completely continuous.

The necessity of property L follows from known properties of propagation of high frequency signals along rays.

Theorem 10 implies that $Z(t)$ has a standard discrete spectrum. There can be no eigenvalue of absolute value one since this would correspond to a solution of the wave equation which is a purely imaginary exponential in time, and according to a theorem of Rellich there are no such solutions with finite energy. Thus the spectral radius of $Z(t)$ is less than one; by the Gelfand formula we conclude that $\|Z(t)\|$ decays exponentially. Thus Theorem 10 gives another proof of the result of [9].

Similar reasoning gives the following result: let f be any element of K, $\sum a_j f_j$ its formal Fourier expansion in terms of the eigenfunctions of $Z(t)$; then

$$\sum a_j e^{\mu_j t} f_j$$

is an asymptotic expansion for $Z(t)f$.

Next we wish to characterize the eigenvalues and eigenfunctions of the generator B of $Z(t)$. For this purpose we say that a solution of the reduced wave equation

$$\Delta u - \mu^2 u = 0 \tag{3.8}$$

in the exterior domain is *outgoing* if the free space solution of the wave equation with initial data $f = [u, -\mu u]$, in symbols $U_0(t)f$, vanishes for $|x| < t - \rho$ for all $t > \rho$. This notion is equivalent with the Sommerfeld definition of outgoing when μ is imaginary. Moreover for arbitrary μ in the case $n = 3$ such a solution of the reduced wave equation can be represented as

$$u(x) = \frac{1}{4\pi} \int_\Gamma \left(u \frac{\partial u}{\partial n} - v \frac{\partial u}{\partial n} \right) dS_y$$

where $v = e^{\mu r}/r$, $r = |x - y|$, and Γ is any smooth surface containing the obstacle but not containing x. The converse is also true.

THEOREM 11. *μ is an eigenvalue of the generator of $Z(t)$ if and only if there exists an outgoing solution of the reduced wave equation* (3.8) *satisfying the boundary conditions.*

SKETCH OF PROOF. We consider $Z_a(t) = P_+^a U(t) P_-^a$ as $a \to \infty$; in the limit this is simply $U(t)$. According to Theorem 4 the eigenvalues μ of the generator of $Z_a(t)$ are simply related to the zeros of the scattering operator; thus they are independent of $a \geqq \rho$. The eigenfunctions depend upon "a" but according to Theorem 6 in a rather trivial fashion. In fact for $b > a \geqq \rho$,

$$f^a = P_+^a f^b. \tag{3.9}$$

Since P_+^a does not alter the data inside the sphere $|x| < a$, it follows that $f^a(x) = f^b(x)$ for $|x| < a$. This shows that the limit

$$\lim_{a \to \infty} f^a(x) \equiv f(x)$$

exists.

Each f_a satisfies

$$Z_a(t) f_a = e^{-\mu t} f_a.$$

Since $Z_a(t)f = U(t)f$ for $|x| < a$, f_a is a solution of the reduced wave equation there. So for $|x| < a$, f_a is of the form

$$f_a = (u_a, \ -\mu u_a),$$

u_a a solution of the reduced wave equation

$$\Delta u_a - \mu^2 u_a = 0$$

which is zero on the obstacle. Since $u_a(x) = u_b(x)$ for $|x| < a$, the limit

$$\lim_{a \to \infty} u_a = u$$

exists. u is in the exterior a solution of the reduced wave equation and is zero on the obstacle. The data $f = [u, \mu u]$ can be thought of as a generalized eigenfunction of $U(t)$; not only does it not lie in H, but it blows up exponentially in $|x|$. f^ρ is orthogonal to D_- and so is f^a for all $a \geqq \rho$ by (3.9). As a consequence the free space solution $U_0(t)f$ vanishes in $|x| < t - \rho$ for $t > \rho$ so that u is outgoing.

Conversely if u is an outgoing solution of the reduced wave equation (3.8) satisfying the boundary conditions, then $e^{-\mu t}$ is an eigenvalue of $Z(t)$. To prove this one shows that the free space translation representation of $f = [u, -\mu u]$ is of the form

$$h(s, \omega) = \begin{cases} 0, & s < -\rho, \\ n(\omega) e^{\mu s}, & s > \rho. \end{cases}$$

Setting

$$h_a(s, \omega) = \begin{cases} h(s, \omega), & s < a, \\ 0, & s > a, \end{cases}$$

one proves that h_a is the free space translation representation of the eigenfunction of $Z_a(t)$ corresponding to the eigenvalue $e^{-\mu t}$.

The above ideas can also be employed to obtain an explicit description of the incoming and outgoing spectral representations of $U(t)$ from which we will in turn be able to obtain an explicit formula for the scattering operator $\mathcal{S}(z)$. We shall denote by $\hat{f}_0$, $\hat{f}_-$, and $\hat{f}_+$ the free space, incoming, and outgoing spectral representations respectively of a given initial data f.

These spectral representations are given by scalar products of f with certain improper eigenfunctions of $U_0(t)$, respectively $U(t)$. We shall show that these improper eigenfunctions consist of exponentials plus certain incoming and outgoing solutions. We recall that the free space spectral representation for $U_0(t)$ is simultaneously incoming and outgoing. Thus D_-^0 and D_+^0 map onto $A_-(N)$ and $A_+(N)$ respectively, while D_- and D_+ map onto $e^{-i\rho z}A_-(N)$ and $e^{i\rho z}A_+(N)$ respectively. We shall limit our considerations to the case $n=3$.

THEOREM 12 (SPECTRAL REPRESENTATION FOR $U_0(t)$).

$$\hat{f}_0(z, \omega) = (f, \phi_0(\cdot, z, \omega)) \tag{3.10}$$

where (,) *denotes the* H_0 *inner product and*

$$4\pi^{3/2}\phi_0(x, z, \omega) = [e^{-izx\cdot\omega}, iz\, e^{-izx\cdot\omega}].$$

The main tool employed in the derivation of (3.10) is the Fourier transform.

THEOREM 13 (INCOMING AND OUTGOING SPECTRAL REPRESENTATIONS FOR $U(t)$). *Let* $v_+(v_-)$ *be the outgoing (incoming) solution of the reduced wave equation*

$$\Delta v + z^2 v = 0$$

satisfying $v + e^{-izx\cdot\omega} = 0$ *on the obstacle. Set*

$$4\pi^{3/2}\psi_\pm(x, z, \omega) = [v_\pm(x, z, \omega), izv_\pm(x, z, \omega)]$$

and define

$$\phi_\pm = \phi_0 + \psi_\pm.$$

Then

$$\hat{f}_\pm(z, \omega) = (f, \phi_\mp(\cdot, z, \omega)), \tag{3.11}$$

where the (,) *denotes the inner product in* H. *Note the switch in signs.*

SKETCH OF PROOF FOR THE INCOMING REPRESENTATION FORMULA. *Step one.* To verify (3.11) for data in D_-. It is required that data in D_- have the same representation as in the free space spectral representation. This in turn requires that $(f, \psi_+) = 0$ for all f in D_-. Now for $f = U_0(-\tau-\rho)w$ where w has support in $|x| < \tau$, it is clear that

$$(f, \psi_+) = (w, U_0(\tau + \rho)\psi_+) = 0$$

since $U_0(\tau+\rho)\psi_+$ vanishes for $|x| < \tau$. It is proved in [9] that linear combinations of such f are dense in D_- and hence (3.11) shares with (3.10) the property of being an isometry in D_-.

Step two. Extend the isometric property of the representation to all translates of D_-. A simple integration by parts shows for any f in D_A that

$$\frac{d}{dt}\hat{f}_-(t) = iz\hat{f}_-(t),$$

where $f(t) = U(t)f$. As a consequence

$$\hat{f}_-(t) = e^{izt}\hat{f}_-.$$

This extends the isometry of the map to all of the translates of D_- and hence to all of H since the translates of D_- are dense in H (see [8]). It also follows that $U(t)$ is represented as multiplication by e^{izt} in this representation.

Step three. The map $f \to \hat{f}_-$ is onto $L_2(-\infty, \infty; N)$. In the case of the free space representation of $U_0(t)$ it is known that the translates of D_- fill out $L_2(-\infty, \infty; N)$ in the representation space. Since D_- and translation are represented by the same objects in both the free space and incoming spectral representations, it follows that the map $f \to \hat{f}_-$ is onto.

THEOREM 14. *The scattering operator is given by*

$$\begin{aligned} \hat{f}_+(z, \omega) &= [\mathcal{S}(z)\hat{f}_-(z, \cdot)](\omega) \\ &= \hat{f}_-(z, \omega) - 2(2\pi)^{1/2} iz \int_{|\theta|=1} s(-\theta, \omega, z)^* \hat{f}_-(z, \theta)d\theta, \end{aligned} \tag{3.12}$$

where

$$\psi_-(r\xi, z, \omega) \sim r^{-1}e^{izr} s(\xi, \omega, z) \quad as \ r \to \infty.$$

SKETCH OF PROOF. It suffices to determine the behavior of $\mathcal{S}(z)$ on D_- since S commutes with $U(t)$ and since translates of D_- are dense