symmetric space $V$ (irreducible or not) is again Hermitan symmetric and is isomorphic to $V$.

**Proof.** Since $H^1(V, \theta) = 0$ [2], we see that the set of points $t \in B$ for which $V_t$ is isomorphic to $V$ is an open set in $B$ [3]; it is also closed by Theorem 2.

**References**


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**ON THE BEST APPROXIMATION FOR SINGULAR INTEGRALS BY LAPLACE-TRANSFORM METHODS**

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1. **Introduction.** Let $f(t)$ be a Lebesgue-integrable function in $(0, R)$ for every positive $R$. We denote by

$$J_\rho(t) = \rho \int_0^t f(t-u)k(\rho u)du$$

$a general singular integral with parameter $\rho > 0$ and kernel $k$ having the following property (P): $k(u) \geq 0$ in $0 \leq u < \infty$, $k \in L(0, \infty)$, and $\int_0^\infty k(u)du = 1$.

If we restrict the class of functions $f(t)$ such that $e^{-ct}f \in L_p(0, \infty)$, $1 \leq p < \infty$, for every $c > 0$, and if $k$ satisfies (P), then the following statements hold:

(i) $J_\rho(t)$ exists as a function of $t$ almost everywhere, $e^{-ct}J_\rho \in L_p(0, \infty)$ for every $c > 0$, and $\|e^{-ct}J_\rho\|_{L_p(0, \infty)} \leq \|e^{-ct}f\|_{L_p(0, \infty)}$;

(ii) $\lim_{\rho \uparrow \infty} \|e^{-ct}\{f - J_\rho\}\|_p = 0$.

Furthermore, we denote by

$$\hat{f}(s) = \int_0^\infty e^{-st}f(t)dt$$

$(s = \sigma + i\tau, \Re \sigma > 0)$

the Laplace-transformation of a function $f$ belonging to one of the classes described above, and the Laplace-Stieltjes-transform of a
function $h(t)$ locally of bounded variation at each $t \geq 0$ with $\int_0^c e^{-ct} |dh(t)| < \infty$ for every $c > 0$ by

$$\mathfrak{h}(s) = \int_0^\infty e^{-st} dh(t) \quad (\text{Re } s > 0).$$

Some fundamental hypotheses upon the kernel $k$ are needed to prove the approximation theorems stated below. Let $\hat{k}(s)$ (Re $s \geq 0$) be the Laplace-transform of $k$: At first,

$$(1.1) \quad \lim_{\rho \uparrow \infty} (s/\rho)^{-\gamma} [1 - \hat{k}(s/\rho)] = A \quad (\text{Re } s > 0)$$

should exist for some real $0 < \gamma \leq 1$, where $A$ is a positive finite constant; secondly, there exists a normalized function $Q(u)$ of bounded variation in $[0, \infty]$ with $Q(\infty) = 1$ such that

$$(1.2) \quad A^{-1} (s/\rho)^{-\gamma} [1 - \hat{k}(s/\rho)] = \tilde{Q}(s/\rho) \quad (\text{Re } s \geq 0);$$

and thirdly, let there be a $q \in L(0, \infty)$, $\int_0^\infty q(u) du = 1$, and

$$(1.3) \quad A^{-1} (s/\rho)^{-\gamma} [1 - \hat{k}(s/\rho)] = \tilde{q}(s/\rho) \quad (\text{Re } s \geq 0).$$

It may be mentioned here that the conditions (1.2) and (1.3), respectively, imply (1.1), but the inverse does not seem to hold. Moreover, if the kernel $k$ is not positive, then the constant $\gamma$ need not be bounded by one.

2. Approximation theorems.

Theorem 1. Let $e^{-ctf}$, $e^{-ctl} \in L(0, \infty)$ for every $c > 0$, let $k$ satisfy (P), and let (1.1) hold for some real $\gamma$ ($0 < \gamma \leq 1$).

(i) Then $||e^{-ct} \{\rho^s (f - J_\rho) - l\}||_{L_1(0, \infty)} = o(1) \quad (\rho \uparrow \infty)$ implies

$$As^\gamma f(s) = l(s) \quad (\text{Re } s > 0)$$

or

$$f(t) = \frac{1}{A} \int_0^t \frac{(t - u)^{\gamma - 1}}{\Gamma(\gamma)} l(u) du \quad \text{a.e.}$$

(ii) If $||e^{-ct} \{f - J_\rho\}||_{L_1(0, \infty)} = O(\rho^{-\gamma}) \quad (\rho \uparrow \infty)$, then there exists a function $F(t)$ locally of bounded variation at $t \geq 0$ with $\int_0^\infty e^{-ct} \left| dF(t) \right| < \infty$ for every $c > 0$ such that

$$(2.1) \quad As^\gamma f(s) = \tilde{F}(s) \quad (\text{Re } s > 0).$$

Sketch of Proof. As (i) can readily be shown, we will restrict ourselves to the proof of (ii). Clearly,
\[ [1 - k(s/p)]f(s) = \int_0^\infty e^{st}\{f(t) - J_\rho(t)\} dt \quad (\text{Re } s > 0), \]
and if we define
\[ S_T(t) = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \left(1 - \frac{|\tau|}{T}\right) e^{st}[1 - k(s/p)]f(s) ds \]
\[ (s = c + it, c > 0), \]
then with the aid of the above equation it can be rewritten as
\[ S_T(t) = \frac{2}{\pi T} \int_0^\infty e^{c(t-u)} \frac{\sin^2 \{T(t-u)/2\}}{(t-u)^2} \{f(u) - J_\rho(u)\} du. \]

Now, the large \(O\)-approximation of \(f\) by \(J_\rho\) gives
\[ \|e^{-ct}S_T\|_{L_1(-\infty, \infty)} \leq \|e^{-ct}\{f - J_\rho\}\|_{L_1(0, \infty)} = O(\rho^{-\gamma}) \quad (\rho \uparrow \infty) \]
for all \(T \geq 0\), and using the condition (1.1) and Lebesgue’s dominated convergence theorem we have
\[ \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \left(1 - \frac{|\tau|}{T}\right) e^{st}Asf(s) ds = \lim_{\rho \uparrow \infty} \rho^nS_T(t), \]
finally, with Fatou's lemma
\[ \|e^{-ct} - \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \left(1 - \frac{|\tau|}{T}\right) e^{st}Asf(s) ds \|_{L_1(-\infty, \infty)} \leq \liminf_{\rho \uparrow \infty} \rho^n\|e^{-ct}S_T\|_{L_1(-\infty, \infty)} = O(1) \]
for all \(T \geq 0\). Evidently the assumptions of the theorem and (1.1) give that \(|Asf(s)|\) is uniformly bounded in \(\text{Re } s \geq \delta > 0\). Now, using a representation theorem for Laplace-transforms [1], this implies the existence of a function \(F(t)\) such that (2.1) is valid.

The condition (2.1) defines a certain class \(K\) of functions \(f\), and Theorem 1 shows, if there exists a constant \(\gamma\) such that (1.1) holds, and if \(\|e^{-ct}\{f - J_\rho\}\|_{L_1(0, \infty)} = O(\rho^{-\gamma})\), then \(f \in K\). The next theorem now shows that the inverse holds too.

**Theorem 2.** Let \(e^{-ct}f \in L(0, \infty)\) for every \(c > 0\), let \(k\) satisfy (\(P\)), and let the relation (1.2) be satisfied for some \(0 < \gamma \leq 1\). Then the assumption (2.1) implies \(\|e^{-ct}\{f - J_\rho\}\|_{L_1(0, \infty)} = O(\rho^{-\gamma})\) \((\rho \uparrow \infty)\).

If the two foregoing theorems hold, we say the singular integral \(J_\rho\) is said to be saturated with order \(O(\rho^{-\gamma})\), and the functions \(f\) yielding
(2.1) define the saturation class of $J_p$. It was J. Favard [5] who introduced this terminology in approximation theory, and one of the authors [2; 3] first made use of Fourier-transform methods to determine the saturation classes of singular integrals which are convolution integrals connected with the Fourier-transform. This question was independently but a little later treated by G. Sunouchi [6] too. Now in this paper general singular integrals are discussed which are classical convolution integrals connected with the Laplace-transform. Although there are connections between the Fourier- and Laplace-transform methods, it may be mentioned that the special properties and peculiar structure of the Laplace-transform play an important role in the proofs and the formulations of the stated theorems.

In the space $L_p(0, \infty), 1 < p < \infty$, an equivalent theorem holds too.

**Theorem 3.** Let $k$ satisfy (P), and let the condition (1.3) exist for some constant $\gamma (0 < \gamma \leq 1)$. A necessary and sufficient condition that the singular integral $J_p(t) = \rho \int_0^1 f(t-u)k(\rho u)du (p > 0)$ shall be saturated with order $O(p^{-\gamma})$ for functions $e^{-ct}f \in L_p(0, \infty), 1 < p < \infty, c > 0$, is that there exists a function $e^{-ct}F \in L_p(0, \infty), c > 0$, such that

$$A s\hat{f}(s) = \hat{F}(s) \quad (\text{Re } s > 0)$$

or

$$f(t) = \frac{1}{A} \int_0^t \frac{(t-u)^{\gamma-1}}{\Gamma(\gamma)} F(u)du \quad a.e.$$

3. Application. As an application we will consider a boundary value problem of heat conduction of a semi-infinite rod ($x \geq 0$). $U(x, t)$ is the temperature in the rod at time $t > 0$, which is described by the equations

$$\frac{\partial U(x, t)}{\partial t} = \frac{\partial^2 U(x, t)}{\partial x^2} (x, t > 0); \quad \lim_{x \to 0} U(x, t) = U_0(t) \quad (t > 0).$$

Among others, G. Doetsch [4, Bd. III] has shown that the solution is given by

$$U(x, t) = \frac{x}{2\sqrt{\pi}} \int_0^t U_0(t-u) \frac{\exp(-x^2/4u)}{u^{3/2}} du \quad (x, t > 0),$$

where $U_0$ is a Lebesgue-integrable function, and that the solution is unique, if the convergence of $U(x, t)$ to $U_0(t)$ is defined by the norm-convergence of the given function space.

$U(x, t)$ is a singular convolution integral with parameter $\rho = 1/x^2 > 0$ and kernel
It is easy to see that the kernel $k$ has property (P) and its Laplace-transform $k(s) = \exp(-\sqrt{s})$ satisfies the conditions (1.1), (1.2), and (1.3) for $\sigma = 1/2$ with $A = 1$.

If we restrict the temperature at the origin $U_0(t)$ such that $e^{-ct}U_0 \in L_p(0, \infty)$, $1 < p < \infty$, $c > 0$, for instance, then making use of Theorem 3 we have:

(i) $\|e^{-ct}\{U_0(t) - U(x, t)\}\|_{L_p(0, \infty)} = o(x)$ ($c > 0$, $x \downarrow 0$) implies $U_0(t) = 0$ a.e.;

(ii) $\|e^{-ct}\{U_0(t) - U(x, t)\}\|_{L_p(0, \infty)} = O(x)$ ($c > 0$, $x \downarrow 0$) guarantees that the flux of heat at the boundary $W_0(t)$ exists a.e., $e^{-ct}W_0 \in L_p(0, \infty)$ for every $c > 0$, and

$$\sqrt{\pi}U_0(s) = W_0(s)$$

(Re $s > 0$)

or, equivalently,

$$U_0(t) = \frac{1}{\sqrt{\pi}} \int_0^t \frac{W_0(u)}{\sqrt{t - u}} \, du \quad \text{a.e.,}$$

and vice versa.

The complete proofs of these and further results as well as a detailed discussion will appear elsewhere (see [7]).

REFERENCES


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