CONVERGENCE OF SEQUENCES OF CONVEX SETS, CONES AND FUNCTIONS

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1. **Introduction and summary.** We shall consider sequences of subsets of Euclidean m-space, especially convex sets that are not necessarily bounded. A notion of convergence of a sequence of sets to a set is introduced, having the property that convergence of convex sets implies convergence of the corresponding polars, and, under a mild condition, of the corresponding projecting cones. It is, however, not true that convergence of convex sets is equivalent to pointwise convergence of the corresponding support functions. Only after introduction of a new type of convergence of a sequence of functions to a function (termed *infimal convergence*) is the desired equivalence achieved. The relation between a sequence of closed convex functions and the sequence of their conjugates is studied. It turns out that either sequence converges infimally if and only if the other does. Finally, infimal convergence of closed convex functions implies convergence of their level sets, under a mild condition. Most of the theorems are valid in more general topological spaces, and sequences may be replaced by nets throughout. Proofs, lemmas and additional theorems will appear elsewhere.

2. **Definitions and notation.** Let O denote the origin. If $x \neq O$, a ray from O through x is denoted (x). A *cone* is a union of rays. In the space of rays a metric can be introduced by identifying each ray with its intersection with the unit $(m-1)$-sphere, and taking (for instance) the chord distance topology on the $(m-1)$-sphere. This defines *open cone*, etc. The *projecting cone* of a set X is $P(X) = \{(x) : x \in X\}$. The *asymptotic cone* of X is $A(X) = \{(x) : (x) = \lim (x_n), x_n \in X, |x_n| \to \infty\}$, where $|\ |$ denotes Euclidean norm. The *distance function* $d(X)$ of X is defined by $d(X, x) = \inf\{|x-y| : y \in X\}$. The *support function* $h(X)$ of X is defined by $h(X, \xi) = \sup_{x \in X} \xi \cdot x$, where $\cdot$ denotes inner product. D_R is the m-disk of radius R. Closure of a set X is denoted by $\overline{X}$ or by $\mathrm{Cl}[X]$.

3. **Convergence of sets and projecting cones.** If X_n, X are sets, we shall define $X_n \to X$ if $d(X_n) \to d(X)$ pointwise. For closed limit sets this corresponds to a definition given by Frolík [3] for general topological spaces. If the sets are closed, it is possible to introduce a

metric agreeing with the above definition of convergence, by generalizing the Hausdorff metric for bounded closed sets.

THEOREM 1. *Let* $\{X_n\}$ *be a sequence of convex sets, and* $X_n \to X$, *then for every open cone* $C \supset A(X)$ *there exist* R *and* N *such that* $X_n \subset C \cup D_R$ *for all* $n > N$.

THEOREM 2. *Let* $\{X_n\}$ *be a sequence of convex sets,* $X_n \to X$, *and* $O \notin \overline{X}$, *then* $P(X_n) \to P(X)$.

A proof of Theorem 2 will appear at the end of this paper.

4. **Support functions and conjugate functions.** It may be hoped that if X_n is convex, and $X_n \to X$, then $h(X_n) \to h(X)$. However, this is not so. In order to obtain the desired implication, and also the converse, we introduce a new type of convergence of functions. If f is a real-valued function, and $\rho > 0$, define ${}_\rho f(x) = \inf\{f(y): |y-x| < \rho\}$.

DEFINITION. *We shall say that* $\{f_n\}$ *converges infimally to* f, *written* $f_n \to_{\inf} f$, *if*

$$\lim_{\rho \to 0} \liminf_{n \to \infty} {}_\rho f_n = \lim_{\rho \to 0} \limsup_{n \to \infty} {}_\rho f_n = f.$$

THEOREM 3. *Let* X_n, X *be convex, then* $X_n \to X$ *if and only if* $h(X_n) \to_{\inf} h(X)$.

Following Fenchel [2] we call a convex function f *closed* if $\lim_{\rho \to 0} {}_\rho f = f$ (for a convex function this is the same as lower semi-continuity). If f is convex and closed, denote $[X, f] = \{(x, a): x \in X, a \geqq f(x)\}$, where $X = \{x: f(x) < \infty\}$. Then $[X, f]$ is a closed convex subset of $(m+1)$-space.

THEOREM 4. $[X_n, f_n] \to [X, f]$ *if and only if* $f_n \to_{\inf} f$.

If f is convex and closed, X defined as above, the *conjugate function* ϕ of f is defined [2] by $\phi(\xi) = \sup_{x \in X}(\xi \cdot x - f(x))$. It is the support function of $[X, f]$ at $(\xi, -1)$. Define $\Xi = \{\xi: \phi(\xi) < \infty\}$, then $[\Xi, \phi]$ is called conjugate to $[X, f]$. The relation of being conjugate is reciprocal.

THEOREM 5. *If the* f_n *and* ϕ_n *are convex and closed, then* $f_n \to_{\inf} f$, *if and only if* $\phi_n \to_{\inf} \phi$. *Therefore, using Theorem* 4, $[X_n, f_n] \to [X, f]$ *if and only if* $[\Xi_n, \phi_n] \to [\Xi, \phi]$.

5. **Polars and level sets.** Let f be a real valued function then for every real number a we define the *level set* $L_a(f) = \{x: f(x) \leqq a\}$. We write $\inf f$ for $\inf f(x)$, where the infimum is taken over all x in the domain of f.

THEOREM 6. *If f_n is convex and closed, and $a > \inf f$, then $f_n \to_{\inf} f$ implies $L_a(f_n) \to L_a(f)$.*

The *polar* (or dual) [1] of X is defined as $X^* = \{\xi : \xi \cdot x \leqq 1$ for all $x \in X\} = L_1(h(X))$.

THEOREM 7. *If X_n is convex, then $X_n \to X$ implies $X_n^* \to X^*$.*

PROOF OF THEOREM 2. For any cone C, $\{\xi : \xi \cdot x \leqq 1$ for all $x \in C\} = \{\xi : \xi \cdot x \leqq 0$ for all $x \in C\}$, so that $C^* = L_0(h(C))$. Put $\mathrm{Cl}[P(X)] = C$, $\mathrm{Cl}[P(X_n)] = C_n$, it is sufficient to show $C_n \to C$. Since C_n and C are convex and closed, and contain O, we have $C_n^{**} = C_n$, $C^{**} = C$ [1], so that by Theorem 7 it is sufficient to prove $C_n^* \to C^*$. Now $\xi \cdot x \leqq 0$ for all $x \in P(X)$ if and only if $\xi \cdot x \leqq 0$ for all $x \in X$ so that $C^* = L_0(h(X))$. Similarly, $C_n^* = L_0(h(X_n))$. Using Theorem 3 we have $h(X_n) \to_{\inf} h(X)$. In Theorem 6 take f_n, f to be $h(X_n)$, $h(X)$, respectively, and take $a = 0 > \inf h(X)$ (since $O \notin \overline{X}$). We conclude $L_0(h(X_n)) \to L_0(h(X))$.

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3. Z. Frolík, *Concerning topological convergence of sets*, Czechoslovak Math. J. **10** (1960), 168–180.

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