treatment given here for product measures in R^2 is based on 2-dimensional interval functions, and seems to the reviewer to be distinctly more complicated than the usual treatment. XI. Derivatives and integrals. Vitali's covering theorem, differentiability of functions of bounded variation, etc., are treated.

No one can quarrel with Professor Hildebrandt's intentions. Analysis should deal with concrete analytic entities, and one can easily lose sight of this in the morass of essential superior integrals, locally negligible sets, etc., of the most abstract treatments of integration theory. It is also true that many U.S. mathematicians [not merely graduate students] are unable to carry out manipulations that any competent Oxford undergraduate could do with ease. But the many avatars of Riemann-Stieltjes integration provide no remedy for this intellectual disease. The Daniell approach to integration, based upon Riemann-Stieltjes integrals for *continuous* functions, carries one quickly and easily to Lebesgue-Stieltjes integrals, and thence to all of the measure theory one can want, in any number of dimensions. Armed with this formidable weapon, one can then attack as many concrete problems as one wishes—everything from pointwise summability of Fourier series in many variables to partial differential equations.

Professor Hildebrandt's book displays the author's mastery of the field, which he has known and contributed to nearly from its beginning. In the reviewer's opinion, however, his book represents the history, and not the future, of integration theory.

EDWIN HEWITT

Diophantine geometry. By Serge Lang. Interscience Tracts in Pure and Applied Mathematics, No. 11. John Wiley and Sons, Inc., New York, 1962. 1+170 pp. $7.45.

Let $(x)=(x_1, x_2, \cdots, x_n)$ be n variables in a given ring R. Let $f(x)=f(x_1, x_2, \cdots, x_n)$ be a polynomial with coefficients in a given field F. We suppose to begin with that R and F are sets of algebraic numbers. The term "Diophantine Analysis" may be applied to the two following topics:

(1) Diophantine Equations, and here there is the question of discussing the solution in R of $f(x)=0$.

(2) Diophantine Approximation, and now the problem is to investigate the lower bound of $|f(x)|$ for all (x) in R.

This book is concerned with probably the three most important developments in the period 1909–1963. There is to begin with the so-called Mordell-Weil Theorem. In 1922, the reviewer showed that if

$n=2$, and both R and F are the rational field, and if $f(x_1, x_2)$ is a cubic polynomial such that the curve $f(x_1, x_2)=0$ is of genus 1, so that the curve has no double point, then a composition process can be given whereby an Abelian group can be associated with the rational points of the cubic, and this group has a finite basis. In simpler geometric terms, this meant that all the rational points can be derived from a finite number by the familiar tangent and chord process. A simpler and more perspicuous proof was given by Weil. He then extended the result to systems of p points on curves of genus p where R and F are now arbitrary algebraic number fields.

Next, the Thue-Siegel-Roth Theorem. In 1909, Thue proved that if θ is an algebraic number of degree n, then the inequality $|q\theta - p| < 1/q^{n/2+\delta}$ has only a finite number of rational integer solutions in p, q for given $\delta > 0$. This result was improved by Siegel, who showed that the exponent $n/2+\delta$ can be replaced by $2\sqrt{n}$. Finally Roth showed in 1955 that the $n/2$ could be replaced by 1, a best possible result.

Lastly, there is Siegel's Theorem. In 1929, he proved that when R is the ring of algebraic integers in an algebraic number field F, then $f(x_1, x_2)=0$ has only a finite number of solutions except in some simple well-defined cases.

For many problems and for a long time, the chief centre of interest was in the cases when R is either the rational field Q or the rational integers Z, and F is Q. Obvious generalizations suggested themselves to more general rings and fields, for example, finite fields, function fields, to the new absolute values associated with valuation theory, and also to several polynomials instead of one.

Various methods have been applied in the past to the problems; arithmetical, algebraical, analytical and geometrical. The necessary geometric ideas were on the whole of such a character that they could be considered algebraic in nature. In recent times, powerful new geometric ideas and methods have been developed by means of which important new arithmetical theorems and related results have been found and proved and some of these are not easily proved otherwise. Further, there has been a tendency to clothe the old results, their extensions, and proofs in the new geometrical language. Sometimes, however, the full implications of results are best described in a geometrical setting. Lang has these aspects very much in mind in this book, and seems to miss no opportunity for geometric presentation. This accounts for his title "Diophantine Geometry." As indicated above, the three main developments are essentially of an arithmetic or algebraic nature. This also applies to the fourth topic treated by

Lang, namely, Hilbert's Irreducibility Theorems, of which the simplest case is as follows. Let $f(x, y)$ be a polynomial with coefficients in a field F, and suppose that $f(x, y)$ is irreducible in F. Are there sets of values of y in F for which the polynomial remains irreducible in x? This is so, for example, if F is a rational field, but is not so if F is the field of complex numbers.

A general question that immediately suggests itself to a reader is what object an author has in mind when writing a book. Some have the true teacher's spirit or even a missionary spirit, wishing to introduce their subject to a wide circle of readers in the most attractive way. Such an author's treatment is essentially self-contained, the presentation is made as simple and complete as possible, and there is no undue generalization that would tend to make unnecessarily difficult the comprehension of the simpler and really fundamental cases; he is painstaking in his efforts to save the reader unnecessary and troublesome effort. When the subject makes undue demands on the reader, the author tries to give the reader some idea of the proof in easily understood language.

Lang is not such an author.[1] Much of the book is practically unreadable unless one is familiar with, among others, Bourbaki, the author's books on Algebraic Geometry and Abelian Varieties, and Weil's Foundations of Algebraic Geometry, and is prepared occasionally to go to the original sources for proofs of some theorems needed in the present volume. Lang may take the point of view that he is only interested in such readers and caters for no others. In fact, he says in his Foreword, "Diophantine problems represent some of the strongest attractions to algebraic geometry." However, in his pages on prerequisites, he refers to the elementary nature of a number of his chapters and their self-containment. Many readers will not accept either of these statements. The topics brought together in this volume are of the greatest interest to a far wider class of readers than those he seems to have in mind. It is unfortunate that it will be exceedingly difficult for them to learn something about most of these topics from the presentation given in this book.

The author's style and exposition leave a great deal to be desired. The results in the book appear as theorems, propositions, properties, lemmas, and even a criterion. The logical distinction between these is

[1] In his recent book on Calculus, he states, " . . . One writes an advanced monograph for one's self, because one wants to give permanent form to one's vision of some beautiful part of mathematics, not otherwise accessible . . . ." Compare with Chaucer's clerk (the scholar and teacher of those days) of whom Chaucer says "and gladly would he learn and gladly teach."

not all all clear. When a reference is made to one of them, the reader must turn over the pages of the relevant chapter to find it. Occasionally it is non-existent, as when on p. 126, Lang refers to Corollary of Theorem 6 of Chapter VI, §5, and on p. 72 to Proposition 7 of Chapter IV, §4, and on p. 148 to Proposition 1 of §1 when he means Corollary 1 of Theorem 1.

His presentation of proofs often seems unattractive. He frequently starts off with various definitions and then gives some arguments and finally says that we have now proved Theorem X, repeating again many of his definitions, sometimes adding a few lines to finish the proof of Theorem X. How much better it would be if he started by stating his theorem and then gave the proof! He would avoid much unnecessary repetition that also makes the enunciation of many results of inordinate length. He would save a great deal of space which could be used most advantageously otherwise.

For example, in Chapter IV, he starts off by saying, "Throughout this chapter, F is a field with a proper set of absolute values M_F satisfying the product formula." At the end of the page, he repeats, "Let F be a field, etc.," and does it again in Lemma 1. It would have been very helpful if he always stated theorems explicitly instead of referring to them by the name of their originators, and he could easily have found room for this.

There is often an impression of vagueness about some of his proofs. They seem to lack the clarity one associates with demonstrations. He deals unsatisfactorily with details. His attitude to them is given elsewhere where he says that in matters of this kind, it is customary to omit details and that he proposes to follow this custom. This will suffice if a writer is giving a sketch of a proof. However, if a result is supposed to have been proved, then it seems desirable that the proof should be complete and presented in a logical manner, e.g., follow the pattern, if A implies B and B implies C, then A implies C. Too often do we see in this book remarks such as something is obvious, or clear, or that it follows, or that it can be proved in the well-known manner, or that the reader will immediately verify, or that it is left as an exercise, or that the proof can be found elsewhere.

Reading is made more difficult because the author gives a far greater number of definitions than is usual with other writers, and it is not easy to retain these in mind. In some chapters, he introduces without explanation technical terms and concepts which are not really necessary for the demonstration and which could be easily dispensed with. He sometimes takes it for granted that when he refers to a theorem by name, the reader will be familiar with what he has in mind. Lang is a very learned mathematician—he oozes mathematics—and assumes that his readers are au fait with all the entities hurled at them. Whenever possible all the resources of algebraic geometry are brought into the proofs of theorems and their generalizations. He is often not content with the ordinary language used in expressing simple ideas, but uses a language of his own which many readers may find difficult to understand. Further he seems to use a method of infinite ascent in expounding his proofs, that is, simple ideas are often developed by using more complicated ones.

Let us now look at the reaction of that would-be reader not too familiar with algebraic geometry, whom Lang possibly did not have in mind but who nevertheless would like to learn something about "Diophantine Geometry," and the important theorems brought together in the present book. The title of the book suggests to him that here is an opportunity for learning something really worth while. He will soon be disillusioned and be faced with a titanic struggle. He will require the patience of Job, the courage of Achilles, and the strength of Hercules to understand the proofs of some of the essential theorems. He will realize that some of the proofs will be above his head, but at any rate he may hope to get some idea of what is being done. He would have been helped enormously if Lang had given a glossary containing definitions of many terms that he uses. Many of these are independent of algebraic geometry. One may begin by mentioning terms which will be familiar to many readers, but not to others, such as: "injective," "bijective" and "surjective"—because in his "Algebraic Geometry," written a few years before, he thinks it necessary to define "surjective." Then there are "finitely generated field" "morphism," "regular extensions of a field," "extension of a field k free from K over k_1," "extension of k which is independent of K over k." Naturally there are a large number of geometric terms, and their definitions may sometimes be found in his other two books from their indexes, but not always. Among these are "transport of structure," "supp ϕ," "isogeny," "fibering," "compatibility of intersections and reduction," "inclusion," "ample." Some of the terms can be defined quite simply, and what a blessing for many readers it would have been if this had been done in the present volume!

Though he gives a table of notation at the end of the book, there are many other symbols which the reader is supposed to know from the author's geometry or to guess at. Though he defines the sets Q, Z of rational numbers and of integers, he does not deem it necessary to define R as the real number field when he first uses the symbol. The meaning of symbols, such as $\nu(nd)$, and the association $A \cdot B$ is taken for granted or is obscure, and some of his geometric formulae might be puzzling. It would be helpful to define a commutative diagram.

Even when he does give definitions or details, he often gives them in the wrong place and at the wrong time, and when they are not really necessary. On p. 65, he refers to "two linearly equivalent divisors" and he first defines the term on p. 131. He reminds his readers of the "meaning of the simple symbol A_K as the group of rational points in K in the commutative group variety A," something one could hardly forget, and also that "P and P' are congruent mod TB_K" means "$P-P' \in TB_K$." He finds it necessary to inform his readers of the definition of a discrete lattice. Distrusting their mathematical powers, he finds it desirable to prove that $\sqrt{(1+x)} < 1+x/2$ since in a bracket, he says "square both sides."

Let us now examine the contents of the book. The author states that Chapters I, II give a good part of classical number theory. Chapter I deals with absolute values on a field. He states that his treatment is much influenced by Artin. The reader would be advised to read Artin since Lang's treatment is not self-contained, and is sometimes lacking in precision. Thus in §§2 on "completions," he states that a field K is complete if every Cauchy sequence converges, omitting the condition that the limit must be in K. He is also under the impression that if $x_n \not\to 0$, then there exists a number $a>0$ such that $|x_n|>a$ for all sufficiently large n. Chapter II deals with the product formula for absolute values, ideals, divisors, units and the class number. He calls these two chapters "elementary." This does not mean that they are easy to read. Here the author begins with his numerous definitions. We have "well-behaved absolute value," "proper sets of absolute values," definitions which many other writers do not find necessary; and then references to schemes, fiber, and various learned references e.g. Noether's normalization theorem, Krull's principal ideal theorem. His examples 2 and 3 cannot be called elementary.

Chapters III and IV deal with the heights of points and also with what have been called the heights of polynomials. The simplest properties are given in Chapter III. The extension to heights on abelian groups leads to the main idea involved in the method of infinite descent, a process first applied by Fermat and of fundamental importance not only in the study of Diophantine Equations but which also has other applications. Chapter IV is concerned with inequalities connecting heights of points and their mappings—in particular by means of linear systems—and the results are of great importance for the main theorems.

Let us now turn to the main theorems of the book—following the sequence given in the book—first, the so-called Mordell-Weil Theorem. Its early history has been already mentioned and there has been a great deal of work done since on the subject. It should be stated again that the only part played by Mordell was in his theorem of some 40 years ago. The Mordell-Weil theorem states, "Let K be a finitely generated field over the prime field. Let A be an abelian variety defined over K. Then the points of K lying in A are finitely generated." In this form the theorem is of considerable generality and so makes great demands upon the technical knowledge of the reader. These are increased since he also shows that the ideas in the proof suffice to prove "the Theorem of the Base."

Let us note some of the concepts required in the chapter. There are a "$K|k$ trace of A," a "Theorem of Chow," "Chow's Regularity Theorem," "Chow Coordinates," "compatibility of projections with specialization," "blowing up a point," "Albanese Variety," "Picard Variety," "Jacobian of a curve," "Chow's theory of the $k(u)|k$-trace." When proof of an extension makes it exceedingly difficult to understand the simpler cases, it might sometimes be better if the generalizations were left in the Journals. The reviewer is reminded of Rip Van Winkle, who went to sleep for a hundred years and woke up to a state of affairs and a civilization (and perhaps a language) completely different from that to which he had been accustomed. There were, however, some things still familiar to him—which is more than can be said by the reviewer about the presentation of the present treatment.

Next, the Thue-Siegel-Roth Theorem. This is presented with the utmost generality. The variables are now elements of an algebraic number field K, and the absolute values are those associated with the field. The author claims to follow Roth's proof. The reader might prefer to read this which requires only a knowledge of elementary algebra and then he need not be troubled with axioms which are very weak forms of the Riemann-Roch Theorem.

Thirdly, Siegel's Theorem. There is given an account of the modern version of Siegel's proof in which θ-functions are replaced by the Jacobian, and Roth's Theorem is used instead of Thue's Theorem. It also gives the extension of the theorem to the case in which the solution is extended to subrings of a field K of finite type over Q.

The final section is on Hilbert's Irreducibility Theorem. Siegel, in his great memoir on Diophantine Approximation, had given an application of his results to this theorem. Lang gives not only the usual elementary proof for the irreducibility over the rational numbers, but also considerable extensions and generalizations. He finds it necessary to introduce into his account Zariski open sets without defining them.

So much for the detailed criticism. This may tend to blind us to some merits of the book. One must mention that the topics have been dealt with systematically so as to show and emphasize algebraic structure and the significance of algebraic geometry in some of the deepest and most important aspects of number theory. One will not be surprised if some of the geometers who read this book will seek further acquaintance with the queen of mathematics.

In conclusion, the reader will need no convincing that Lang, as has already been said, is a very learned mathematician, thoroughly familiar with every aspect of the topics he deals with, and their developments. His interesting and valuable historical notes give further evidence of this. Lang assumes that his readers are as knowledgeable as he is, and can grapple with the subject with the same ease that he does. Even if they could, Lang's style is not such as to make matters easy for them. Lang in writing is not a follower of Gauss, whose motto was "pauca sed matura." Further thought and care about his book, before publication, would have been well worth while. Those who can understand the book will be indebted to him for having brought together in one volume the important results contained in it. How much greater thanks would he have earned if the book had been written in such a way that more of it could have been more easily comprehended by a larger class of readers! It is to be hoped that some one will undertake the task of writing such a book.

L. J. MORDELL