ON BOUNDING HARMONIC FUNCTIONS
BY LINEAR INTERPOLATION

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It is well known [1], [4] that Poisson’s formula for the value at the
origin O of a function which is harmonic inside a circle $(x-x_0)^2 + (y-y_0)^2 = A^2$ can be written in the form

$$u(O) = \frac{1}{2\pi} \int_0^{2\pi} \frac{R(\theta + \pi)u(R(\theta), \theta) + R(\theta)u(R(\theta + \pi), \theta + \pi)}{R(\theta) + R(\theta + \pi)} \, d\theta,$$

where $r = R(\theta)$ is the polar equation of the boundary. Thus the value of a harmonic function at any point in a circle is an average of the values obtained by linear interpolation of the boundary values at the ends of each chord through the point.

In particular, it follows that

$$u(O) \leq \max R(\theta + \pi)u(R(\theta), \theta) + R(\theta)u(R(\theta + \pi), \theta + \pi) \quad \frac{R(\theta) + R(\theta + \pi)}{R(\theta) + R(\theta + \pi)}.$$

It is tempting to conjecture that a similar inequality holds for harmonic functions in any convex or even star-shaped domain. Recently J. Barta [2], [3] has given two incomplete proofs of this conjecture.

We shall show that in general no inequality of the form

$$u(O) \leq M \max R(\theta + \pi)u(R(\theta), \theta) + R(\theta)u(R(\theta + \pi), \theta + \pi) \quad \frac{R(\theta) + R(\theta + \pi)}{R(\theta) + R(\theta + \pi)}$$

(1)

can hold for all harmonic functions in a star-shaped domain $r < R(\theta)$. In fact, an inequality of the form (1) holds for each point $O$ of a convex domain $D$ only if $D$ is the interior of a circle.

We first prove:

**Lemma.** Let $G$ be the Green’s function with singularity at $O$ for the
two-dimensional domain $D: r < R(\theta)$. An inequality of the form (1) holds for all harmonic functions $u$ if and only if the identity

$$R(\theta)(R(\theta)^2 + R'(\theta)^2)^{1/2} \frac{\partial G}{\partial n} (R(\theta), \theta)$$

(2)

$$= R(\theta + \pi)(R(\theta + \pi)^2 + R'(\theta + \pi)^2)^{1/2} \frac{\partial G}{\partial n} (R(\theta + \pi), \theta + \pi)$$

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holds for all $\theta$. If (2) is satisfied, (1) holds with $M = 1$.

**Proof.** Let $a$ be a constant such that

$$\frac{R(\theta + \pi)u(R(\theta), \theta) + R(\theta)u(R(\theta + \pi), \theta + \pi)}{R(\theta) + R(\theta + \pi)} \leq a$$

for all $\theta$.

We write the representation

$$u(O) = -\int_{r=R(\theta)} u \frac{\partial G}{\partial n} ds$$

in the form

$$u(O) = a - \int_0^{2\pi} [u(R(\theta), \theta) - a] \frac{\partial G}{\partial n} (R(\theta), \theta) \frac{ds}{d\theta} d\theta.$$  

In the identity

$$\int_0^{2\pi} f(\theta) g(\theta) d\theta = \frac{1}{2} \int_0^{2\pi} \left\{ [f(\theta) + f(\theta + \pi)] [g(\theta) + g(\theta + \pi)] + [f(\theta) - f(\theta + \pi)] [g(\theta) - g(\theta + \pi)] \right\} d\theta$$

we let

$$f(\theta) = \frac{u(R(\theta), \theta) - a}{R(\theta)},$$

$$g(\theta) = -R(\theta) \frac{\partial G}{\partial n} (R(\theta), \theta) \frac{ds}{d\theta}$$

$$= -R(\theta) (R(\theta)^2 + R'(\theta)^2)^{1/2} \frac{\partial G}{\partial n} (R(\theta), \theta).$$

Then $g(\theta) \geq 0$. By (3),

$$f(\theta) + f(\theta + \pi) \leq 0.$$  

Thus,

$$u(O) \leq a + \frac{1}{2} \int_0^{\pi} [f(\theta) - f(\theta + \pi)] [g(\theta) - g(\theta + \pi)] d\theta.$$  

Equality holds if $f(\theta) + f(\theta + \pi) = 0$; that is, for those boundary values $u(R(\theta), \theta)$ satisfying

$$R(\theta + \pi)u(R(\theta), \theta) + R(\theta)u(R(\theta + \pi), \theta + \pi) = a[R(\theta) + R(\theta + \pi)].$$
If \( f(\theta) \) is made to satisfy only this condition, the function \( f(\theta) - f(\theta + \pi) \) is completely arbitrary for \( 0 \leq \theta < \pi \). The right-hand side of (8) and therefore also \( u(0) \) can be made arbitrarily large unless \( g(\theta) - g(\theta + \pi) = 0 \). This is the condition (2).

If (2) is satisfied, (8) becomes \( u(0) \leq a \), which is (1) with \( M = 1 \).

We remark that (2) is certainly satisfied if the symmetry condition

\[
R(\theta + \pi) = R(\theta)
\]

holds. This means that the point \( O \) bisects each chord through it. This is true at the center of an ellipse, or of a parallelogram. In such a case we find that

\[
u(0) \leq \max \frac{1}{2}[u(R(\theta), \theta) + u(R(\theta + \pi), \theta + \pi)].
\]

We can now prove:

**Theorem.** If a bound of the form (1) for harmonic functions \( u \) holds at each point \( O \) of a convex domain \( D \) with smooth boundary \( C \), then \( C \) is a circle.

**Proof.** We consider the chord \( PQ \) connecting any two boundary points \( P \) and \( Q \). Let its length be \( d \), and let \( O \) be the point on this chord at distance \( \delta \) from \( Q \).

Let the chord make angles \( \alpha \) and \( \beta \), respectively, with the normals at \( P \) and \( Q \).

By hypothesis, (1) holds at \( O \). Hence by the lemma we have

\[
(9) \quad \frac{(d - \delta)^2}{\cos \alpha} \frac{\partial G}{\partial n}(O, P) = \frac{\delta^2}{\cos \beta} \frac{\partial G}{\partial n}(O, Q).
\]

We let \( O \) approach \( Q \) by making \( \delta \to 0 \). It is easily seen that
\[ \frac{\partial G}{\partial n} (O, Q) = \frac{-1}{\pi \delta} \cos \beta + O(1). \]

(The leading term comes from Green's function for the half-plane.)

On the other hand, since \( (\partial G/\partial n)(O, P) = 0 \) for \( O \) on \( C \),

\[ \frac{\partial}{\partial n} G (O, P) = -\cos \beta \frac{\partial^2 G}{\partial n_P \partial n_Q} (P, Q) + O(\delta^2). \]

Dividing (9) by \( \delta \) and letting \( \delta \to 0 \), we find

\[ \frac{d^2}{dn_P dn_Q} G (P, Q) \cos \beta \cos \alpha = \frac{1}{\pi}. \]

The function \( \partial^2 G/\partial n_P \partial n_Q \) is symmetric in \( P \) and \( Q \). Letting \( \delta \to d \),

we find the same equation with \( \alpha \) and \( \beta \) interchanged. Hence \( \cos \alpha = \cos \beta \). This is true for all \( P \) and \( Q \) on \( C \). Letting \( Q \to P \) on \( C \) and

using the fact that \( \beta \) is a continuous function of \( Q \), we find that \( \alpha = \beta \).

An elementary exercise in differential geometry shows that \( \alpha = \beta \)

for all \( P \) and \( Q \) on \( C \) implies that \( C \) is a circle. This proves the theorem.

**Remark.** If we restrict our attention to non-negative \( u \):

\[ u(R(\theta), \theta) \geq 0, \]

the inequality (8) does lead to a bound of the form (1) with the best possible constant

\[ M = 1 + \int_0^\pi \max \left\{ \frac{1}{R(\theta + \pi)} \left[ g(\theta) - g(\theta + \pi) \right], \frac{1}{R(\theta)} \left[ g(\theta + \pi) - g(\theta) \right] \right\} d\theta. \]

(10)

However, the evaluation of this constant requires rather detailed information about the kernel \( \partial G/\partial n \), which is difficult to come by.

In this case the maximum principle gives (1) with

\[ M = 1 + \max_{0 \leq \theta \leq 2\pi} \left\{ \frac{R(\theta + \pi)}{R(\theta)} \right\}, \]

which is just what one obtains by means of crude estimates for the Green’s function in (10).

The analogous results in \( n \) dimensions can be proved in the same manner.
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A NOTE ON THE FUNDAMENTAL THEORY OF ORDINARY DIFFERENTIAL EQUATIONS

BY GEORGE R. SELL

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In this note we present some results on various problems connected with ordinary differential equations which do not necessarily satisfy a uniqueness condition. Using the concept of an integral funnel we are able to generalize the classical theorem on continuity with respect to initial conditions. This then leads to a reformulation of the problem of classifying the solutions of a given differential equation. That is, it is shown that every continuous vector field \( f(x) \) on \( W \) gives rise to a bicontinuous injection of \( W \) into a space of functions \( \mathcal{H} \), and consequently the problem of classifying solutions is equivalent to the problem of characterizing this family of bicontinuous injections. A detailed discussion, with proofs, will appear later.

1. Introduction. Let us consider the differential equation

\[
x' = f(x)
\]

where \( f \) is defined and continuous on some open, connected set \( W \) in \( \mathbb{R}^n \), real \( n \)-space. We shall let \( W^* = W \cup \{\omega\} \) denote the one-point compactification of \( W \). There is then at least one solution \( \phi(p, t) \) of (1) through every point \( p \in W \) with \( \phi(p, 0) = p \). Moreover, every solution is defined on some maximal interval \( J_p \) where either \( J_p = \mathbb{R}^1 \) or \( \phi(p, t) \to \{\omega\} \) as \( t \to \text{bdy } J_p \). It should be noted that since the solutions of (1) may not be unique, the interval \( J_p \) depends not only on \( p \).

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