NORM DECREASING HOMOMORPHISMS OF GROUP ALGEBRAS[1]

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Communicated by G. A. Hedlund, February 10, 1964

1. **Introduction.** Let G be a locally compact group, let $C_0(G)$ be the Banach space of all continuous complex-valued functions on G which vanish at infinity, and let $M(G)=C_0(G)^*$. In [2] Glicksberg has identified all subgroups of the unit ball $\Sigma_{M(G)}$ in $M(G)$ for locally compact abelian groups G, and all positive subgroups of $\Sigma_{M(G)}$ when G is a compact group. In a subsequent paper [3], he used these results to give the structure of norm decreasing homomorphisms ϕ: $L^1(F) \to M(G)$ in the situations when (1) F is locally compact and G is locally compact *abelian*, or (2) F is locally compact, G is *compact*, and $\mu \geqq 0 \Rightarrow \phi(\mu) \geqq 0$ for $\mu \in L^1(F)$. The results given here identify all subgroups of $\Sigma_{M(G)}$ for any locally compact group G, and give the structure of all norm decreasing homomorphisms ϕ: $L^1(F) \to M(G)$ where F and G are locally compact groups. At the same time we show that every homomorphism of this type has a natural extension to a norm decreasing homomorphism ϕ: $M(F) \to M(G)$.

2. **Subgroups of the unit ball in** $M(G)$. The structure theorems are based on the following observations about measures in $M(G)$. If $\mu \in M(G)$ we let $s(\mu)$ denote the support of μ and let $|\mu|$ denote the total variation of μ [1, p. 122]. The norm of $\mu \in M(G)$ is indicated by $\|\mu\|$.

THEOREM 1. *If G is a locally compact group and if $\mu, \lambda \in M(G)$ are such that $\|\mu * \lambda\| = \|\mu\| \cdot \|\lambda\|$, then*

(1) $s(\mu * \lambda) = (s(\mu)s(\lambda))^-$,

(2) $|\mu * \lambda| = |\mu| * |\lambda|$.

In Glicksberg [2], similar results are proved in limited situations: (1) is proved for positive measures, and (2) is proved for measures in a subgroup of $\Sigma_{M(G)}$ when G is compact.

If K is a compact subgroup of a locally compact group G, we denote normalized Haar measure on K as m_K. If $K^\wedge$ is the collection of continuous unimodular multiplicative functions on K and if $\rho \in K^\wedge$, then the measure ρm_K is defined such that $\int_G \psi d\rho m_K = \int_K \psi \cdot \rho dm_K$ for $\psi \in C_0(G)$.

[1] This research was partially supported by Air Force Office of Scientific Research Grant No. AFOSR-407-63.

THEOREM 2. *If G is a locally compact group and if $\mu \in M(G)$ is such that $\mu * \mu = \mu$ and $\|\mu\| = 1$, then there is a compact subgroup $K \subset G$ and a function $\rho \in K^\wedge$ such that $\mu = \rho m_K$.*

This is an easy consequence of Theorems 1 and 2. It was also proved by Glicksberg in [2] for compact G, and by Loynes [4] for locally compact G if $\mu \geqq 0$. This theorem gives the structure of the unit in any subgroup of $\Sigma_{M(G)}$ for locally compact G. Next we give the complete structure of such subgroups, and then go on to identify all subgroups of $\Sigma_{M(G)}$.

THEOREM 3. *If Γ is a subgroup of $\Sigma_{M(G)}$ for locally compact group G, then $H_0 = \bigcup \{s(\mu) : \mu \in \Gamma\}$ is a subgroup of G and the unit of Γ is of the form $i = \rho m_K$ where K is a compact subgroup of G, $K \subset H_0$, K is normal in H_0, and $\rho \in K^\wedge$. Furthermore, there is a subgroup $\Omega \subset S \times G$ such that $H_0 = \{g \in G : (\alpha, g) \in \Omega$ for some $|\alpha| = 1\}$ and $\Gamma = \{\alpha\delta_g * \rho m_K : (\alpha, g) \in \Omega\}$.*

Here S is the circle group, $S \times G$ is the usual product group, and if $g \in G$ we let δ_g denote the point mass concentrated at g.

PROPOSITION 4. *If G is a locally compact group and if Γ is a subgroup of $\Sigma_{M(G)}$, let $H_0 = \bigcup \{s(\mu) : \mu \in \Gamma\}$ and write the unit of Γ as $i = \rho m_K$ where $K \subset H_0$ is a compact subgroup of G which is normal in H_0, and $\rho \in K^\wedge$. Then $K_0 = \mathrm{Ker}(\rho)$ is a compact subgroup of G which is normal in both H_0 and K. Furthermore, K is central in H_0 (mod K_0) so that $K/K_0 \subset \mathrm{cent}(H_0/K_0)$.*

This property is necessary since we must have $\rho m_K * \delta_g * \rho m_K = i * (\delta_g * \rho m_K) = \delta_g * \rho m_K$ for all $g \in H_0$ if Γ is to have the structure given in Theorem 3 and be closed under convolution with $i = \rho m_K$ is an identity. Now the subgroups of $\Sigma_{M(G)}$ can be catalogued explicitly.

THEOREM 5. *Let G be a locally compact group and let Γ be a subgroup of $\Sigma_{M(G)}$ with unit $i = \rho m_K$. Then we have the following structural properties:*

(1) *$H_0 = \bigcup \{s(\mu) : \mu \in \Gamma\}$ is a (not necessarily closed) subgroup of G.*
(2) *$i = \rho m_K$ where K is a compact subgroup of G and $\rho \in K^\wedge$.*
(3) *K and $K_0 = \mathrm{Ker}(\rho)$ lie within H_0 and are both normal in H_0.*
(4) *K is central in H_0 (mod K_0).*
(5) *$\Omega = \{(\alpha, g) \in S \times G : \alpha\delta_g * \rho m_K \in \Gamma\}$ is a subgroup of $S \times G$ with $H_0 = \{g \in G : (\alpha, g) \in \Omega$ for some $|\alpha| = 1\}$, and we have $\Gamma = \{\alpha\delta_g * \rho m_K : (\alpha, g) \in \Omega\}$.*

Conversely, let H_0 be any subgroup of G, let K be a compact subgroup *of G lying within H_0, and let $\rho \in K^\wedge$ be chosen such that*

(1) *K and $K_0 = \mathrm{Ker}(\rho)$ are both normal in H_0,*

(2) *K is central in H_0 (mod K_0),*

and let Ω be any subgroup of $S \times G$ with $H_0 = \{g \in G: (\alpha, g) \in \Omega$ for some $|\alpha| = 1\}$. Then $\Gamma = \{\alpha\delta_g * \rho m_K: (\alpha, g) \in \Omega\}$ is a subgroup of $\Sigma_{M(G)}$ with $H_0 = \bigcup\{s(\mu): \mu \in \Gamma\}$, with $i = \rho m_K$ as a unit, and with $\Omega = \{(\alpha, g) \in S \times G: \alpha\delta_g * \rho m_K \in \Gamma\}$.

3. **Norm decreasing homomorphisms of group algebras.** Let F and G be locally compact groups and let $\phi: L^1(F) \to M(G)$ be a norm decreasing homomorphism. Consider on $M(G)$ the (σ)-topology (weak $*$ topology) and on $M(F)$ the (so)-topology, which is defined to be the strong operator topology obtained by letting $M(F)$ act by left convolution on the two-sided ideal $L^1(F)$. The structure theory for ϕ can be derived from the analysis of the subgroups of $\Sigma_{M(G)}$ because of the continuity properties of ϕ exhibited in the following theorem. Recall that the extreme points of $\Sigma_{M(F)}$ have (so)-dense convex span in $\Sigma_{M(F)}$.

THEOREM 6. *Let F and G be locally compact groups and let $\phi: L^1(F) \to M(G)$ be any norm decreasing homomorphism. Then there is an extension $\Phi: M(F) \to M(G)$ which is a norm decreasing homomorphism and is continuous on norm bounded sets as a mapping $\Phi: (M(F), (so)) \to (M(G), (\sigma))$. If $\{e_j: j \in J\}$ is a left approximate identity of norm one in $L^1(F)$ then the extension is given explicitly by the formula*

$$\Phi(\mu) = \lim\{\phi(e_j * \mu): j \in J\}$$

for $\mu \in M(F)$, where this limit is taken in the (σ)-topology.

If G is a locally compact group and we are given a subgroup Γ in $\Sigma_{M(G)}$, let us define $\tau: S \times G \to \Sigma_{M(G)}$ such that $\tau(\alpha, g) = \alpha\delta_g * \rho m_K$, where $i = \rho m_K$ is the unit of Γ, K is a compact subgroup of G, and $\rho \in K^\wedge$. Then let us define

$$\Omega = \{(\alpha, g) \in S \times G: \alpha\delta_g * \rho m_K \in \Gamma\},$$

$$\Omega_0 = \{(\alpha, g) \in S \times G: \alpha\delta_g * \rho m_K = i\}.$$

Then $\tau: \Omega \to \Gamma$ is a continuous onto homomorphism with $\mathrm{Ker}(\tau) = \Omega_0$, so that Ω_0 is normal in Ω. We also define the canonical map $\pi: S \times G \to (S \times G/\Omega_0)_r$ (the right coset space), which is a homomorphism onto Ω/Ω_0 when restricted to Ω.

Now let $\phi: L^1(F) \to M(G)$ be a norm decreasing homomorphism with extension $\Phi: M(F) \to M(G)$ as in Theorem 6, and with $\Gamma = \Phi(\mathcal{E}_{M(F)})$

where $\mathcal{E}_{M(F)} = \{\delta_x : x \in F\}$ in $\Sigma_{M(F)}$. Let us consider the map $\theta: F \to \Omega/\Omega_0$ defined such that $\theta(x) = \pi \circ \tau^{-1} \circ \bar{\phi}(\delta_x)$ for all $x \in F$.

PROPOSITION 7. *The map $\theta: F \to \Omega/\Omega_0$ is an onto homomorphism and is continuous as a mapping $\theta: F \to (S \times G/\Omega_0)_r$.*

We obtain an integral representation of ϕ by taking $\psi \in C_0(G)$ and forming $\tilde{\psi}: (\alpha, g) \to \langle \alpha\delta_g * \rho m_K, \psi \rangle$, so that $\tilde{\psi} \in C_0(S \times G)$ and is constant on right cosets of Ω_0. Then we have $\tau^*\psi = {}^{\sim} \circ \pi^{-1} \in C_0((S \times G/\Omega_0)_r)$ and we have the following structure theorem.

THEOREM 8. *Let F and G be locally compact groups and let $\phi: L^1(F) \to M(G)$ be a norm decreasing homomorphism which is not identically zero. Then if $\bar{\phi}: M(F) \to M(G)$ is its extension to $M(F)$ and if $\Gamma = \bar{\phi}(\mathcal{E}_{M(F)})$, we have the relation*

$$(*) \qquad \langle \bar{\phi}(\mu), \psi \rangle = \langle \mu, \tau^*\psi \circ \theta \rangle$$

for all $\mu \in M(F)$ and $\psi \in C_0(G)$.

Conversely, if Γ is a given subgroup of $\Sigma_{M(G)}$ and if $\theta: F \to \Omega/\Omega_0$ is a continuous onto homomorphism, then the relation $(*)$ defines a norm decreasing homomorphism $\bar{\phi}$ on $M(F)$ which is continuous on norm bounded sets as a mapping $\bar{\phi}: (M(F), (so)) \to (M(G), (\sigma))$, and we have $\Gamma = \bar{\phi}(\mathcal{E}_{M(F)})$.

More elegant structure theorems are available if $\phi: L^1(F) \to M(G)$ is a norm decreasing isomorphism. These results, together with proofs of the preceding theorems, will be published at a later date.

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