ON THE LOCAL BEHAVIOR OF THE RATIONAL
TSCHEBYSCHEFF OPERATOR

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Let \( l \) and \( r \) be non-negative integers. Denote by \( \mathcal{R}_{l,r} \) the set of all rational functions where the degrees of the numerator and denominator do not exceed \( l \) and \( r \) respectively. If \( R = \frac{p}{q} \in \mathcal{R}_{l,r} \) and \( p \) and \( q \) are relatively prime polynomials of degree \( \deg p \) and \( \deg q \), then \( d_{l,r}[R] := \min \{ l - \deg p, r - \deg q \} \) is called the defect of \( R \) in \( \mathcal{R}_{l,r} \): the function \( R \) is called degenerate, if the defect is positive. (For these notations compare Werner (1962) [3].)

For a fixed interval \([a, b]\) let \( T_{l,r}[f] \) be the Tschebyscheff Approximation of \( f \in C[a, b] \) in the class \( \mathcal{R}_{l,r} \) with respect to the norm \( \| f \| := \max_{[a, b]} |w(x) \cdot f(x)| \), with \( w(x) \) a positive continuous weight function in \([a, b]\). We write \( \eta_{l,r}[f] := \| f - T_{l,r}[f] \| \). Those \( f \) for which \( T_{l,r}[f] \) is not degenerate are called normal by Cheney and Loeb (1963) [1]. Already Maehly and Witzgall (1960) [2] proved that \( T_{l,r}[f] \) furnishes a continuous map of \( C[a, b] \) into itself at \( f \) with respect to the introduced norm, if \( f \) is normal. For the actual verification of normality one may use the following normality criterion:

Let \( g(x) \) be normal for \( T_{l,r} \). Then \( f(x) \) is normal if

\[
\| f - g \| < \left( \eta_{l-1,r-1}[g] - \eta_{l,r}[g] \right)/2.
\]

Except for the case \( r = 1, l \) arbitrary (compare Werner (1963) [3]) no specific properties of \( f \) are known to insure normality of \( f \) for arbitrary \( l, r \).

Maehly and Witzgall (1960) [2] also gave an example that showed that \( T_{l,r}[f] \) need not be continuous at \( f \), if \( f \) is not normal. Recently Cheney and Loeb (1963) [1] made an extensive study of generalized rational approximation and proved that \( T_{l,r}[f] \) is not continuous, if \( f \) is not normal and if no alternant of the error function \( \eta(x) := w(x)(f(x) - T_{l,r}[f](x)) \) has \( r + l + 2 \) points. This later restriction may be lifted and one obtains the following classification.

**Theorem 1.** The operator \( T_{l,r}[f] \) is continuous at \( f \) if and only if \( f \) is normal or belongs to the class \( \mathcal{R}_{l,r} \).

In order to prove this, one now only has to cope with the case that the error function has an alternant of \( l + r + 2 \) points. By a proper

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\( ^1 \) Added in proof. Recently a criterion has been published by H. L. Loeb, Notices Amer. Math. Soc. 11 (1964), 335.
construction one finds a sequence of continuous functions \( f_n \); \( n = 1, 2, \ldots \) that converges uniformly to \( f \) and whose associated \( T \)-approximations do not converge to \( T_{i,r}[f] \).

The construction is not quite easy, because on the other hand one can prove that \( T_{i,r}[f_n](x) \) converges to \( T_{i,r}[f](x) \) pointwise in \((a, b)\), if \( f_n \) converges to \( f \) uniformly in \([a, b]\), and if \( d_{i,r}[T_{i,r}[f]] \leq 1 \). This result shows that one might expect convergence in a somewhat looser sense. If the defect is greater than 1, then pointwise convergence no longer persists, although from every sequence \( f_n \) uniformly converging to \( f \) a subsequence can be extracted for which the associated \( T \)-approximations converge pointwise with at most \( r \) exceptional points in \([a, b]\). Thus the best one can hope for is convergence in measure.

**Theorem 2.** Given \( f \in C[a, b] \). To every \( \epsilon > 0, \epsilon_1 > 0 \) one can find \( \delta > 0 \) such that

\[
\|f - g\| < \delta
\]

implies that there is a finite number of intervals depending on \( g \) whose total length is less than \( \epsilon_1 \) such that for all points of \([a, b]\) not lying in the said intervals the inequality

\[
|T_{i,r}[f](x) - T_{i,r}[g](x)| < \epsilon
\]

holds.

The proofs of these results will be given elsewhere, the methods used are similar to that of §7 of Werner (1962) [4].

**References**


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