A SPARSE REGULAR SEQUENCE OF EXPONENTIALS
CLOSED ON LARGE SETS

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Introduction. For a given sequence \( \{ \lambda_k \} \) of complex numbers, the problem of determining those intervals \( I \) on which the exponentials \( \{ e^{i\lambda_k x} \} \) are complete in various function spaces has been extensively studied [3]. Since the problem is invariant under a translation of \( I \), only the lengths of \( I \) are involved, and attention has focused on the relation between these lengths and the density of the sequence \( \{ \lambda_k \} \). With the function space taken to be \( L^p(I) \) for \( 1 \leq p < \infty \), or \( C(I) \), the continuous functions on \( I \), the general character of the results has been that there exist sparse real sequences (\( \lim r^{-1}(\text{the number of } |\lambda_k| < r) = 0 \), for example) for which \( I \) can be arbitrarily long [2], but all such sequences are nonuniformly distributed; when a sequence is sufficiently regular, in the sense that \( \lambda_k \) is close enough to \( k \), the length of \( I \) cannot exceed \( 2\pi \) [4, p. 210]. Most recently, in a complete solution which accounts for all these phenomena, Beurling and Malliavin have proved that the supremum of the lengths of \( I \) is proportional to an appropriately defined density of \( \{ \lambda_k \} \) [1].

The purpose of this note is to show that the situation is quite different when the single interval \( I \) is replaced by a union of intervals. Specifically, we will construct a real symmetric (or positive) sequence \( \{ \lambda_k \} \) arbitrarily close to the integers, for which the corresponding exponentials are complete in \( C(S) \), where \( S \) is any finite union of the intervals \( |x - 2n\pi| < \pi - \delta \), with integer \( n \) and \( \delta > 0 \), and so has arbitrarily large measure. Thus, for sets \( S \) more general than intervals,
it would seem that no relation can be expected between measure of $S$ and density of $\{\lambda_k\}$.

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Results.

Lemma 1. We may partition the positive integers into an infinite number of disjoint sequences $S_r = \{k\}^{n}_{n=1}, r = 1, 2, \ldots$, with the property that $\lim sup n/k^{(r)} = 1$ for each $r$.

Proof. We will define $S_r$ as the union $\bigcup_{n=1}^{\infty} \sigma_{i,r}$ of disjoint blocks $\sigma_{i,r}$ of consecutive integers. To define $\sigma_{i,r}$, we order the integer couples $(i, r)$ with $i, r \geq 1$, by increasing values of $s = i+r$, and for same values of $s$, by increasing $i$. We let $\sigma_{1,1} = \{1\}$ and choose the remaining $\sigma_{i,r}$ consecutively in the order of the $(i, r)$, letting each $\sigma_{p,q}$ begin with the first integer not included in the previously defined $\sigma$; we pick $\sigma_{p,q}$ so long that if $N$ is the number of integers in $\sigma_{p,q}$, $k$ is the first of them, and $M$ is the total number of integers in the (already determined) $\sigma_{j,q}$ with $j < p$, then $(N+M)/(k+N-1) > 1-1/p$. By this construction, whenever $k^{(r)}$ in $S_r$ coincides with the right-hand endpoint of a $\sigma_{i,r}$ we have $n/k^{(r)} > 1-1/i$, so that $\lim sup n/k^{(r)} = 1$ for each $r$. Finally, the $S_r$ are all disjoint and their union is all positive integers. Lemma 1 is established.

Lemma 2. With $\theta_1, \theta_2, \ldots$ real numbers, set $z_k = e^{i2\pi \theta_k}$, and denote by $\Delta(\theta_1, \ldots, \theta_n)$ the determinant whose $2j$th row is $z_j^{-1}, z_j^{-2}, \ldots, z_j^{-n}$ and whose $(2j-1)$th row is $z_j^{n+1}, z_j^{n+2}, \ldots, z_j^n$, with $1 \leq j \leq n$. Then given $\epsilon > 0$ we may choose $\theta_1, \theta_2, \ldots$ with $|\theta_i| < \epsilon$ so that, for all $n$, we have $\Delta(\theta_1, \ldots, \theta_n) \neq 0$.

Proof. The condition $|\theta_i| < \epsilon$ is equivalent to $z_i \in \gamma$, with $\gamma$ an appropriate arc of $|z| = 1$. First, letting $z_1$ be any point of $\gamma$ other than $z = 1$ ensures $\Delta(\theta_1) \neq 0$. Then we observe that $\Delta(\theta_1, \ldots, \theta_n)$ can be expanded as a polynomial in $z_n$ and $z_n^{-1}$, with leading coefficient $\Delta(\theta_1, \ldots, \theta_{n-1})$. Assuming $z_1, \ldots, z_{n-1}$ have been chosen to satisfy the requirements of the lemma, this coefficient does not vanish, and so $\Delta(\theta_1, \ldots, \theta_n)$ considered as a function of $z_n$ is not identically zero; being analytic in $z_n$ it therefore cannot vanish everywhere for $z_n$ on $\gamma$. Thus we may find a point $e^{i2\pi \theta_n} \in \gamma$ such that when $z_n = e^{i2\pi \theta_n}$, $\Delta(\theta_1, \ldots, \theta_n) \neq 0$. By induction, Lemma 2 is established.

Theorem. Given $\epsilon > 0$, there exists a symmetric real sequence $\{\lambda_k\}_{k=-\infty}^{\infty}$ with $|\lambda_k - k| < \epsilon$ such that the functions $\{e^{i2\pi \lambda_k}\}$ are complete in continu-
ous functions on every finite union of the intervals $|x-2n\pi|<\pi-\delta$, with integer $n$ and $\delta>0$.

**Proof.** We will partition the integers into disjoint subsets, shift each subset by a small amount, and let the sequence $\{\lambda_k\}$ consist of the points so obtained. Then we will show that completeness of the corresponding exponentials on unions of certain intervals is equivalent to completeness on a single interval of $\{e^{ikx}\}$, with $k$ in one subset, and thereby reduce the theorem to a classical result.

Let $S_r, r=1, 2, \ldots$, be the disjoint subsets of the integers defined in Lemma 1, and let $S_r=\{k\mid -k\in S_r\}$. Similarly, let $\theta_r, r=1, 2, \ldots$, be the numbers constructed in Lemma 2, and let $\theta_r=-\theta_r$. Now for $k\in S_r, r=\pm 1, \pm 2, \ldots$, set $\lambda_k=k+\theta_r$, and $\lambda_0=0$. Then the sequence $\{\lambda_k\}_{k=0}^\infty$ is symmetric and $|\lambda_k-k|<\epsilon$.

To prove the theorem we must show that given $N$ and $\delta>0$, the exponentials $\{e^{\lambda_kx}\}$ are complete in $C(S)$, where $S=\bigcup_{n=-N+1}^N I_n$, and $I_n$ is the interval $|x-2n\pi|<\pi-\delta$, or equivalently [4, p. 115] that any bounded measure supported on $S$ which annihilates these exponentials must vanish identically. Accordingly, let $\mu(x)$ be such a measure, and denote by $\mu_n(x-2n\pi)$ the restriction of $\mu(x)$ to $I_n$. Then $\mu_n(x)$ is a bounded measure supported on $I_0$, and

$$\mu(x) = \sum_{n=-N+1}^N \mu_n(x-2n\pi).$$

Now by a change of variable,

$$\int_S e^{\lambda_kx}d\mu(x) = \sum_{n=-N+1}^N e^{\lambda_k2n\pi} \int_{I_0} e^{\lambda_kx}d\mu_n(x),$$

and if $k\in S_r, e^{\lambda_k2n\pi}=e^{\theta_r2n\pi}$ and does not depend on $k$. Thus if $\mu(x)$ annihilates the exponentials $\{e^{\lambda_kx}\}$ for $k\in S_r$, so does

$$\sum_{n=-N+1}^N e^{i\theta_r2n\pi}\mu_n(x),$$

which is a bounded measure supported on the single interval $I_0$.

We now invoke a known result [3, p. 13]: since by Lemma 1, $\lim sup n/k_n=1$ in each $S_r$, the set $S_r$ has Polya density 1, and so the exponentials $\{e^{ikx}\}$ for $k\in S_r$ are complete in continuous functions on any interval of length less than $2\pi$, in particular on $I_0$. By definition of the set $\{\lambda_k\}$ for $k\in S_r$ as a translate of $S_r$, the same is true of the exponentials $\{e^{\lambda_kx}\}, k\in S_r$, and consequently the measure (2) on $I_0$ which annihilates them must vanish identically. We conclude
for each $r$. Writing (3) with $r = \pm 1, \pm N$ yields a system of $2N$ linear equations for the $2N$ measures $\mu_n(x)$, $-N+1 \leq n \leq N$, whose determinant is precisely $\Delta(\theta_1, \cdots, \theta_N)$ and so, by Lemma 2, does not vanish. Thus the only solution to this system is $\mu_n(x) \equiv 0$, $-N+1 \leq n \leq N$, whence $\mu(x) \equiv 0$ by (1). This completes the proof of the theorem.

**Remarks.** 1. By an obvious modification of the proof, the exponentials $\{e^{ikx}\}$ with $k \in S_r$ for $r > 0$ have the same completeness property.

2. We may give a constructive proof of the theorem along the same lines. Shifting each $I_n$ to $I_0$ transforms the problems of approximating a continuous function on $S$ by linear combinations of the exponentials $\{e^{ikx}\}$ into that of solving a system of linear equations on $I_0$ with nonzero determinant, and thereby again reduces matters to approximating on $I_0$ by linear combinations of $\{e^{ikx}\}$ for $k \in S_r$.

**BIBLIOGRAPHY**


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