THE CLOSING LEMMA AND STRUCTURAL STABILITY

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Introduction. Consider a differentiable n-manifold M. Let $\mathfrak{X}=\mathfrak{X}(M)$ be the space of all C^1 tangent vector fields on M under a C^1 topology [1]. Each $X\in\mathfrak{X}$ induces a C^1-flow on M called the X-flow. Let d be a metric on M and let ϵ be positive. Two flows are homeomorphic if there is a homeomorphism h of M onto itself taking the trajectories of one flow onto those of the other; the two flows are ϵ-homeomorphic if h can be chosen so that $d(h(p), p)<\epsilon$ for all $p\in M$. X is said to be structurally stable if, given $\epsilon>0$, there then exists a neighborhood $\mathfrak{U}$ of X in $\mathfrak{X}$ such that for each $Y\in\mathfrak{U}$ the Y-flow is ϵ-homeomorphic to the X-flow. Let us say that X is crudely structurally stable if we drop the ϵ condition: X is crudely structurally stable if there exists a neighborhood $\mathfrak{U}$ of X in $\mathfrak{X}$ such that $Y\in\mathfrak{U}$ implies that the Y-flow is homeomorphic to the X-flow. Let Σ denote those X in $\mathfrak{X}$ which are structurally stable and let Σ_c denote those X in $\mathfrak{X}$ which are crudely structurally stable, obviously $\Sigma\subset\Sigma_c$. The problem of structural stability theory is to characterize Σ and Σ_c and to study the topological relation of Σ and Σ_c to $\mathfrak{X}$.

The most comprehensive results in structural stability theory are due to M. Peixoto [2], [3], [4] who has shown, when M is a compact 2-manifold, that $\Sigma=\Sigma_c$, $\overline{\Sigma}=\mathfrak{X}$, and that the fields in Σ are characterized completely as the fields with "generic" induced flows.

Related to the problem of structural stability is the following conjecture:

CLOSING LEMMA. *If the X-flow has a nontrivial recurrent trajectory through some $p\in M$ and if $\mathfrak{U}$ is any neighborhood of X in $\mathfrak{X}$ then there exists $Y\in\mathfrak{U}$ such that the Y-flow has a closed orbit through p.*

(Recall that a trajectory is nontrivially recurrent if it is contained in its α- or in its ω-limit set without being a closed orbit or a stationary point.)

Results concerning the Closing Lemma. M. Peixoto [4] has proved the Closing Lemma in the case that M is the 2-torus and X has no

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singularities. We prove the following two forms of the Closing Lemma. (Our proofs, however, are invalid for a C^r topology on $\mathfrak{X}^r$, $r>1$.)

THEOREM 1. *Let M be any differentiable 2-manifold and let $X\in\mathfrak{X}$ have a nontrivial recurrent trajectory through $p\in M$. Let U be an arbitrarily small coordinate neighborhood of p in M and let $\epsilon>0$ be given. Then there exists $\Delta\in\mathfrak{X}$ such that*

(a) *Δ vanishes on $M-U$.*

(b) *The C^1 size of Δ respecting the coordinates of U is less than ϵ.*

(c) *$Y=X+\Delta$ has a closed orbit through p.*

THEOREM 2. *Let M be a compact n-manifold and let a Riemannian metric be put on M so that the norm of each linear transformation $L: T_x(M)\to T_y(M)$ is defined. Suppose that $X\in\mathfrak{X}$ induces a flow ϕ which has a nontrivial recurrent trajectory through $p\in M$. Define $J(t, x)$: $T_x(M)\to T_{\phi(t,x)}(M)$ to be the jacobian isomorphism of tangent spaces induced by $x\to\phi(t, x)$. Suppose that $\epsilon>0$ is given and that*

$$\lim_{t\to\infty}\frac{1}{t}\left\|J^{-1}(t, p)\right\| = 0.$$

Then there exists $\Delta\in\mathfrak{X}$ such that the C^1 size of Δ is less than ϵ and $Y=X+\Delta$ has a closed orbit through p.

Where M is compact, all Riemannian metrics are equivalent and so Theorem 2 does not depend on the choice of Riemannian metric.

DEFINITION. Let X be in $\mathfrak{X}(M)$ for a differentiable n-manifold M. *A flow-box* for X at $p\in M$ is a coordinate neighborhood U of p in M such that in terms of the coordinates $(u^1, \cdots, u^n)$ of U, $u^i(p)=0$ for $i=1, 2, \cdots, n$ and

$$X_u = \left(\frac{\partial}{\partial u^1}\right)_u \qquad \text{for all } u \text{ in } U.$$

If $X_p\neq 0$, then it is well known that a flow-box for X at p exists.

The following lemma is the principal tool used to prove Theorems 1 and 2.

LEMMA. *Let $\epsilon>0$ and $0<\delta<1$ be given. Let M be a differentiable n-manifold and let $X\in\mathfrak{X}$ induce the flow ϕ. Suppose that X does not vanish at $p^*\in M$ and let U be a flow-box for X at p^*. Let*

$$\Pi = \left\{(0, u^2, u^3, \cdots, u^n)\in U\right\}.$$

Suppose that P is a subset of Π such that arbitrarily near p^* there are distinct points of P lying on a common ϕ-trajectory (e.g., let $P=\mu\cap\Pi$

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and let $p^* \in \bar{\mu} \cap \Pi$ *where* μ *is a nontrivial recurrent* ϕ*-trajectory). Then there exist points* p *and* q *of* P *such that*

$$\text{(a)} \qquad \begin{aligned} &|p - p^*| < \epsilon, \\ &|q - p^*| < \epsilon, \\ &\phi(t^*, p) = q \text{ for some } t^* > 0, \end{aligned}$$

and

$$\text{(b)} \qquad \begin{aligned} &\text{If } r = \phi(t', p) \in P \text{ for some } t',\ 0 < t' < t^*, \\ &\text{then } |p - r| > \delta |p - q| \text{ and } |q - r| > \delta |p - q|, \end{aligned}$$

where $|x-y|$ *denotes the distance between* x *and* y *respecting the coordinates of* U.

The proof of this lemma is easy. Just take a p_0 and q_0 in P obeying (a) where ϵ has been replaced by the smaller constant $\frac{1}{2}(1-\delta)\cdot\epsilon$ and where t^* is called t_0. If (b) fails to be true for some $r=\phi(t', p_0)$, then suppose that $|q_0-r| \leqq \delta|p_0-q_0|$. Replace q_0 by r and regard the pair (p_0, r) instead of the pair (p_0, q_0). Call $(p_0, r)=(p_1, q_1)$. Proceed similarly if $|q_0-r|>\delta|p_0-q_0|$ but $|p_0-r| \leqq \delta|p_0-q_0|$ to get $(p_1, q_1)=(r, q_0)$. Proceed with (p_1, q_1) as was done with (p_0, q_0), getting, thereby, a sequence (p_k, q_k) $k=1, 2, \cdots$. The process ends at a finite step (p_m, q_m) because $\phi(t, p)$ crosses Π at most a finite number of times for $0 \leqq t \leqq t_0$. The pair (p_m, q_m) satisfies (b) by construction, It also satisfies (a) because

$$\begin{aligned} |p^* - p_m| &\leqq \sum_{i=1}^{m} \max(|p_i - p_{i-1}|, |q_i - q_{i-1}|) + |p_0 - p^*| \\ &\leqq \sum_{i=1}^{m} \delta^i |p_0 - q_0| + |p_0 - p^*| \\ &< |p_0 - q_0| \cdot \frac{1}{1-\delta} + |p_0 - p^*| \\ &< \frac{\epsilon\cdot(1-\delta)}{2\cdot(1-\delta)} + \frac{\epsilon(1-\delta)}{2} < \epsilon. \end{aligned}$$

Similarly $|p^*-q_m|<\epsilon$.

As a consequence of Theorem 1, M. Peixoto's paper [**4**] can be shortened considerably. The methods used to prove Theorem 1 can also be used to solve the following problem.

Suppose that $M=S^2$, $X\in\mathfrak{X}(S^2)$, and that the X-flow has a closed orbit γ which is isolated but unstable. Suppose there are n generic saddle points $p_1, p_2, \cdots, p_n$ outside γ and n more generic saddle points $q_1, q_2, \cdots, q_n$ inside γ such that one separatrix from each p_i

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has γ as an ω-limit and one separatrix from each q_i has γ as an α-limit point. The problem is to find an arbitrarily C^1 small $\Delta \in \mathfrak{X}$ such that for $Y = X + \Delta$, the Y-flow "joins the p_i's to the q_j's." That is, each p_i should have a Y-separatrix σ_i which is also a Y-separatrix of some q_j. When Δ is sufficiently C^1 small, it is easily seen that the same q_j cannot be joined to two different p_i's. M. Peixoto [**4**] has solved this problem for $n = 1$. The problem for $n \geqq 2$ is related to an investigation of "higher order structural stability" at present being completed by G. Sottomayor. Sottomayor wishes Δ to be C^5 small, but—as in the Closing Lemma itself—our methods only produce perturbations which are C^1 small.

I hope that Theorem 2 will yield as a corollary that *distal* minimal nontrivial recurrent flows on compact differentiable manifolds may be closed by arbitrarily C^1 small perturbations Δ. It would suffice to prove that for some $p \in M$, $\|J^{-1}(t, p)\|$ is bounded as $t \to \infty$ where J is the jacobian isomorphism induced as in Theorem 2. Roughly, the reason this should be true is that $\|J^{-1}\|$ is a measure of how fast the flow contracts and distal flows don't contract too much.

Finally, we inspect two examples related to the theory of structural stability for noncompact 2-manifolds. First we show that for $M = R^2$, $\Sigma_c \neq \Sigma$. Second, following M. L. Peixoto, we see that there exists a nonvanishing $X \in \mathfrak{X}(R^2)$ which is not in Σ_c. This shows that it will probably be quite difficult to characterize the elements of Σ and Σ_c for noncompact 2-manifolds.

In a sense, this is unfortunate because Theorem 1 holds for noncompact differentiable 2-manifolds and one might hope to use it to try to generalize M. Peixoto's characterization theorem [**4**] to the noncompact case. In particular one would hope to show that $X \notin \Sigma_c$ if the X-flow has a nontrivial recurrent trajectory. I can prove this if M has finite genus but if M has infinite genus, I can prove it only by using the following

CONJECTURE. *Suppose that M is a differentiable 2-manifold and that $X \in \Sigma_c(M)$. Let Γ be the union of all the closed orbits of the X-flow. Then Γ is closed in M.*

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