ON THE SYMMETRY OF CONVEX BODIES

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We say that a convex body in \( n \)-dimensional Euclidean space \( E_n \) is "\( k \)-symmetric" if it coincides with its reflection through some \( k \)-plane. Let \( K \) be an \( n \)-dimensional convex body and \( K' \) a \( k \)-symmetric convex body of maximum volume contained in \( K \). Define

\[
c(K; k) = \frac{V(K')}{V(K)},
\]

where \( V(K) \) is the volume of \( K \). Let

\[
c(n, k) = \inf \{ c(K; k) : K \subset E_n \}.
\]

**Theorem 1.**

\[
c(n, k) \geq \max \left\{ k!, (n - k)! \right\} \frac{2^{n-k}}{2^n}, \quad 0 \leq k < n.
\]

This generalizes the result, \( c(n, 0) > 2^{-n} \), proved in [3].

One can also consider \( K \) as a nonhomogeneous solid with density \( f(p) \) at each \( p \in K \), and ask for a symmetric subset of maximum mass. Restricting ourselves to the case of \( 0 \)-symmetry (i.e., central symmetry), we define for each integrable density \( f \) on \( K \)

\[
\mu(K; f) = \frac{M(K')}{M(K)},
\]

where \( K' \) is a centrally symmetric convex body of maximum mass contained in \( K \), and \( M(K) \) is the mass of \( K \). Let \( \mu(K) \) be the infimum of \( \mu(K; f) \), for \( f \) ranging over all integrable densities, and define

\[
\mu(n) = \inf \{ \mu(K) : K \subset E_n \}.
\]

**Theorem 2.** \( \mu(n) \geq 2^{-n}, \quad n \geq 3, \) and \( \mu(2) = 1/3. \)

The first inequality follows from an obvious generalization of the computation of "mean symmetry" used in [3], while the second equality depends on the fact (see Theorem 4) that any plane convex body is the union of 3 centrally symmetric convex bodies.

Let \( g(n) \) be the least number \( r \) such that any \( n \)-dimensional convex body \( K \) can be covered by \( r \) translates of \(-K\) (equivalently, \( g(n) \) is the least number \( r \) such that any \( n \)-dimensional convex body is the
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union of $r$ centrally symmetric bodies). Grünbaum [2] defines the number $h(n)$ as the least number $r$ with the following property: if $\mathcal{F}$ is any family of pairwise intersecting translates of a convex body $K \subset \mathbb{R}^n$, then there exist $r$ points such that each member of $\mathcal{F}$ contains at least one of them.

**Theorem 3.** $h(n) \leq g(n) \leq c(n, 0)^{-1}$, for all $n$.

It is shown in [1] that

$$c(n, 0) < \sqrt{\frac{2}{\pi}} \left(\frac{2}{e}\right)^n \left(\frac{n}{n+1}\right)^{n-1} \sqrt{n+1}.$$  

Together with Theorem 3, this implies that $h(n)$ grows faster than any fixed power of $n$, showing that the conjecture of [2], viz. $h(n) \leq n+1$, is false. Indeed, the conjecture fails for $n = 3$, since $g(3) \geq 7$.

The last inequality follows from the fact that a tetrahedron $T$ in $\mathbb{R}^3$ cannot be covered with fewer than 7 translates of $-T$. In $\mathbb{R}^2$ we have a sharp result.

**Theorem 4.** $g(2) = 3$.

**References**


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