THE RECURSIVE EQUIVALENCE TYPE OF
A CLASS OF SETS

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1. Introduction. Let us consider non-negative integers (numbers),
collections of numbers (sets) and collections of sets (classes). The
letters $\varepsilon$ and $\varnothing$ stand for the set of all numbers and the empty set
of numbers respectively. We write $\subseteq$ for inclusion, proper or improper.
A mapping from a subset of $\varepsilon$ into $\varepsilon$ is called a function; if $f$ is a func-
tion, we denote its domain and its range by $\delta f$ and $\rho f$ respectively. Let
a class of mutually disjoint nonempty sets be called an md-class; such
a class is therefore countable, i.e., finite or denumerable. We recall
that the recursive equivalence type (abbreviated: RET) of a set $\alpha$,
denoted by $\text{Req}(\alpha)$, is defined [1, p. 69] as the class of all sets which
are recursively equivalent to $\alpha$. We wish to consider the problem:
"How can we define the RET of an md-class in a natural manner?"
Throughout this note $S$ stands for an md-class and $\sigma$ for the union of
all sets in $S$; for every $x \in \sigma$ we denote the unique set $\alpha$ such that
$x \in \alpha \subseteq S$ by $\alpha_x$.

DEFINITIONS. A set $\gamma$ is a choice set of $S$, if
(1) $\gamma \subseteq \sigma$,
(2) $\gamma$ has exactly one element in common with each set in $S$.
The set $\gamma$ is a good choice set of $S$ (abbreviated: gc-set), if it also
satisfies
(3) there exists a partial recursive function $p(x)$ such that $\sigma \subseteq \delta p$
and $(\forall x)[x \in \sigma \Rightarrow p(x) \in \gamma \cdot \alpha_x]$.

Consider the special case that the md-class $S$ is a finite class of
finite sets. Then
(a) every choice set of $S$ is a good choice set,
(b) every two choice sets of $S$ are recursively equivalent,
(c) every two good choice sets of $S$ are recursively equivalent.

If the md-class $S$ is infinite, (a) and (b) need no longer be true.
For let $S$ contain infinitely many sets of cardinality $\geq 2$, e.g.,
$S = ((0, 1), (2, 3), (4, 5), \cdots )$. Then $S$ has $\varepsilon$ choice sets. Every good
choice set of $S$ has the form $p(\sigma)$ for some partial recursive function
$p(x)$, hence $S$ has at most $\aleph_0$ good choice sets and (a) is false. Every
nonzero RET contains exactly $\aleph_0$ sets; the $\varepsilon$ choice sets of $S$ can
therefore not all be recursively equivalent and (b) is false. On the

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other hand, (c) still holds. For we have

**Proposition P1.** Every two good choice sets of an md-class are recursively equivalent.

Note that (a) does not even hold for every finite class consisting of two infinite sets. For let \( S = (\tau, \tau') \), where \( \tau \) and \( \tau' \) are complementary immune sets. Then \( S \) has denumerably many choice sets, but if \( S \) had a good choice set, \( \tau \) and \( \tau' \) would be recursive. For every md-class \( S \) we write \( \xi(S) \) for the class of all gc-sets of \( S \). If \( \xi(S) \) is nonempty, \( S \) is called a gc-class. The class \( (\tau, \tau') \) mentioned above is an example of an md-class which is not a gc-class. P1 enables us to give the

**Definition.** For any gc-class \( S \),

\[
\text{RET}(S) = \text{Req}(\gamma), \quad \text{for any } \gamma \in \xi(S).
\]

If \( S \) is a finite md-class of finite sets, \( S \) is a gc-class and \( \text{RET}(S) \) equals the cardinality of \( S \). We need not exclude the trivial case that \( S \) is empty, for then \( \xi(S) \) contains exactly one set, namely \( o \).

2. **Elementary properties.** The sets \( \alpha_0, \ldots, \alpha_n \) are separable if there exist mutually disjoint r.e. sets \( \beta_0, \ldots, \beta_n \) such that \( \alpha_i \subseteq \beta_i \), for \( 0 \leq i \leq n \). We write \( \alpha_0 | \alpha_1 \) if \( \alpha_0 \) and \( \alpha_1 \) are separable.

**Proposition P2.** The finite md-class \( S = (\alpha_0, \ldots, \alpha_n) \) is a gc-class if and only if \( \alpha_0, \ldots, \alpha_n \) are separable; if \( S \) is a gc-class, each choice set of \( S \) is a gc-set and \( \text{RET}(S) \) equals the cardinality of \( S \).

A gc-class is called isolated if each (or equivalently, at least one) of its gc-sets is isolated. In other words, a gc-class is isolated if its RET is an isol. For every nonempty gc-class \( S \) we have: \( \sigma \) is a finite set if and only if \( S \) is a finite class of finite sets. Similarly,

**Proposition P3.** Let \( S \) be a nonempty gc-class. Then \( \sigma \) is an isolated set if and only if \( S \) is an isolated class of isolated sets.

Two classes \( S_1 \) and \( S_2 \) with unions \( \sigma_1 \) and \( \sigma_2 \) respectively are separable if \( \sigma_1 \mid \sigma_2 \). For any two classes \( A \) and \( B \) we write

\[
A \times B = \{j(\alpha \times \beta) \mid \alpha \in A \text{ and } \beta \in B\},
\]

where \( j(x, y) = x + (x+y)(x+y+1)/2 \).

**Proposition P4.** Let \( S_1 \) and \( S_2 \) be separable md-classes. Then \( S_1 \cup S_2 \) is an md-class and

(a) \( S_1 \cup S_2 \) is a gc-class if and only if both \( S_1 \) and \( S_2 \) are gc-classes,

(b) if \( S_1 \cup S_2 \) is a gc-class, \( \text{RET}(S_1 \cup S_2) = \text{RET}(S_1) + \text{RET}(S_2) \).
PROPOSITION P5. Let \( S_1 \) and \( S_2 \) be nonempty md-classes. Then \( S_1 \times S_2 \) is a nonempty md-class and
(a) \( S_1 \times S_2 \) is a gc-class if and only if both \( S_1 \) and \( S_2 \) are gc-classes,
(b) if \( S_1 \times S_2 \) is a gc-class, \( \text{RET}(S_1 \times S_2) = \text{RET}(S_1) \cdot \text{RET}(S_2) \).

3. The class Bin(\( \alpha \)). Let \( \{ \rho_n \} \) be the canonical enumeration of the class of all finite sets \([2, \text{p. 81}] \) and \( r_n = \text{cardinality of } \rho_n \). For any set \( \alpha \) and any number \( k \) we write
\[
C(\alpha, k) = \{ n \mid \rho_n \subset \alpha \text{ and } r_n = k \}, \quad \text{Bin}(\alpha) = \{ C(\alpha, k) \mid k \geq 1 \}.
\]
Note that Bin(\( \alpha \)) is an md-class for any set \( \alpha \); if \( \alpha \) is a finite set of cardinality \( n \), the members of Bin(\( \alpha \)) are separable and Bin(\( \alpha \)) is a gc-class with \( n \) as cardinality and RET. For any infinite set \( \alpha \), Bin(\( \alpha \)) is a denumerable md-class of infinite sets; the next proposition tells us when Bin(\( \alpha \)) is a gc-class. We write \( \text{Req}(\epsilon) = R \) and refer to [2, pp. 80, 84] for the definition of a regressive set and a regressive isol.

PROPOSITION P6. Let \( \alpha \) be infinite and \( A = \text{Req}(\alpha) \). Then
(a) if \( \alpha \) has an infinite r.e. subset, Bin(\( \alpha \)) is a gc-class of RET R,
(b) if \( \alpha \) is a regressive set, Bin(\( \alpha \)) is a gc-class of RET A,
(c) if \( \alpha \) is immune, but not regressive, Bin(\( \alpha \)) is not a gc-class.

It follows that among the \( c \) existing md-classes of immune sets, exactly \( c \) are gc-classes and exactly \( c \) are not. It is shown in [3] that though the collection \( \Lambda_R \) of all regressive isols is not closed under addition one multiplication, one can extend the \( \text{min}(x, y) \) function from \( e^2 \) into \( \epsilon \) in a natural manner to a \( \text{min}(X, Y) \) function from \( \Lambda_R^2 \) into \( \Lambda_R \). However, \( \text{min}(X, Y) \) need no longer assume one of the values \( X \) and \( Y \).

PROPOSITION P7. Let \( \alpha, \beta \) be two nonempty isolated sets, \( A = \text{Req}(\alpha) \)
\( B = \text{Req}(\beta) \) and
\[
S = \{ j(\xi \times \eta) \mid (\exists n)( n \geq 1 \text{ and } \xi = C(\alpha, n) \text{ and } \eta = C(\beta, n) ) \}.
\]
If \( \alpha \) and \( \beta \) are regressive, i.e., \( A, B \subseteq \Lambda_R \) then \( S \) is a gc-class with \( \text{RET}(S) = \text{min}(A, B) \).

It can be shown that \( S \) may be a gc-class while the sets \( \alpha \) and \( \beta \) are immune, but not both regressive.


DEFINITIONS. Let \( p(x) \) be a partial recursive function and \( S \) a gc-class. Then \( p(x) \) is a gc-function of \( S \), if
(\alpha) \sigma \subseteq \delta \rho \text{ and } \rho(\sigma) \subseteq \xi(S),
(\beta) (\forall x) [x \in \sigma \Rightarrow p(x) \subseteq \rho(\sigma) \cdot \alpha_x],
(\gamma) \rho \delta \rho \subseteq \delta \rho \text{ and } (\forall x) [x \in \delta \rho \Rightarrow p^2(x) = p(x)].

A gc-function is a partial recursive function which is a gc-function of at least one gc-class.

Every gc-class has at least one gc-function. For if a partial recursive function \( p(x) \) is related to \( S \) by \((\alpha) \) and \((\beta) \), then \( p(x) \) has a restriction which satisfies \((\alpha) \), \((\beta) \) and \((\gamma) \). With every partial recursive function \( p(x) \) we associate the md-class \( \text{Gen}(\rho) = \{ p^{-1}(y) \mid y \in \rho \delta \rho \} \) of r.e. sets. This md-class is empty if and only if \( p(x) \) is nowhere defined.

**Proposition P8.** A partial recursive function \( p(x) \) is a gc-function if and only if it satisfies \((\gamma) \). Moreover, if \( p(x) \) satisfies \((\gamma) \), it is a gc-function of the class \( S = \text{Gen}(\rho) \) with \( \sigma = \delta \rho \) and \( \rho(\sigma) = \rho \delta \rho \subseteq \xi(S) \).

**Proposition P9.** Let \( p(x) \) be a gc-function of the gc-class \( S \). Then
\[
\delta \rho = \sigma \iff S = \text{Gen}(\rho).
\]

**Definition I.** A class \( S \) is primitive, if it satisfies one of the three conditions: (i) \( S \) is empty, (ii) \( S \) is a nonempty, finite md-class of r.e. sets, (iii) \( S \) is a denumerable md-class of r.e. sets and there exists a recursive function \( a(n, x) \) such that if \( \alpha_n = p a(n, x) \), then \( S \) consists of the distinct sets \( \alpha_0, \alpha_1, \cdots \).

**Definition II.** A class \( S \) is primitive, if it is a gc-class with a gc-function \( p(x) \) such that \( S = \text{Gen}(\rho) \).

**Definition III.** A class \( S \) is primitive, if \( S = \text{Gen}(\rho) \) for some partial recursive function \( p(x) \).

**Proposition P10.** The three definitions of a primitive class are equivalent.

**Corollary.** A class \( S \) is primitive if and only if it is a gc-class with a gc-function \( p(x) \) such that \( \delta \rho = \sigma \).

**Definition.** An md-class \( T \) is a restriction of the gc-class \( S \), if
(a) for every \( \beta \subseteq T \), there is an \( \alpha_0 \) such that \( \beta \subseteq \alpha_0 \subseteq S \),
(b) there is a \( \gamma \subseteq \xi(S) \) such that \( \beta \subseteq T \Rightarrow \gamma \cdot \alpha_0 \subseteq \beta \).

**Proposition P11.** An md-class is a gc-class if and only if it is a restriction of some primitive gc-class.

While there are \( c \) gc-classes, only \( \mathbb{N}_0 \) of them are primitive. For each RET \( A \) there exists a gc-class with \( A \) as its RET, but a primitive class can only have one of 0, 1, \( \cdots \), \( R \) as its RET. The gc-sets of a primitive class \( P \) are readily characterized. For if \( P \) is finite, the gc-sets of \( P \) are the choice sets of \( P \), and if \( P \) is infinite, say.
\[ P = (\alpha_0, \alpha_1, \cdots), \quad \alpha_n = \rho a(n, x), \]

\( a(n, x) \) a recursive function, then \( \gamma \in \xi(p) \) if and only if \( \gamma = \rho a(f_n, u_n) \), for a recursive permutation \( f_n \) and a recursive function \( u_n \). Finally, the restrictions of any given primitive class can be simply described. Thus Proposition P11 serves a purpose.

REFERENCES


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