

A SHARP FORM OF THE VIRIAL THEOREM¹

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In its classical form the Virial Theorem concerns the behavior of a system S of n mass particles acting under Newtonian attraction in such a fashion that the center of mass 0 remains fixed and the potential energy V satisfies $V > -\infty$ for all positive time t . The latter condition, which is not always stated explicitly, guarantees the analyticity of the coordinates of the particles in the independent variable t ; in particular, it excludes collisions [2, pp. 324 ff.]. Let T denote the kinetic energy and h the (constant) total energy $T + V$. Let \hat{V} denote the time average

$$\hat{V} = \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t V(\tau) d\tau$$

if the limit exists, with an analogous definition of \hat{T} . Clearly each of \hat{V} , \hat{T} exists if the other does and $\hat{T} + \hat{V} = h$. The usual theorem states that if S is bounded, in the sense that distances between particles and the velocities of the particles remain bounded, then \hat{T} and \hat{V} exist and $2\hat{T} = -\hat{V}$. An equivalent conclusion is

$$(1) \quad \hat{T} = -h.$$

In this form the theorem is mathematically unsatisfactory because the condition of boundedness is far from necessary. This is already demonstrated by the parabolic case $h=0$ of the two-body problem, $n=2$. In this case distance grows like $t^{2/3}$, so that V behaves like $-t^{-2/3}$ as $t \rightarrow \infty$. Consequently, $\hat{V}=0$. Hence $\hat{T}=0$, which is consistent with (1).

We shall replace boundedness by a condition which is both necessary and sufficient. Let $r_{jk}(t)$ denote the distance between particle j and particle k at time t , and let $R(t) = \max_{j,k} r_{jk}(t)$.

THEOREM 1. (1) is true if and only if

$$(2) \quad R(t) = o(t), \quad t \rightarrow \infty.$$

Let $2I$ denote the moment of inertia of the system with respect to 0 . We begin by showing that (2) is equivalent to

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$$(3) \quad I(t) = o(t^2), \quad t \rightarrow \infty.$$

Since [2, p. 243]

$$(4) \quad 2IM = \sum_{1 \leq j < k \leq n} m_j m_k r_{jk}^2,$$

where M is the total mass, it follows from the definition of R that $I < C_1 R^2$, where C_1 is independent of time. On the other hand, according to (4),

$$2IM > mm' \sum r_{jk}^2,$$

where m, m' are the two smallest masses. At each instant of time $R(t)$ is one of the r_{jk} . Therefore, $2IM > mm'R^2$, so $C_2 R^2 < I$ where C_2 is a positive constant independent of time. Hence

$$C_2 R^2 < I < C_1 R^2,$$

which implies the equivalence of (2) and (3).

From an integration it is clear that the relation

$$(5) \quad \ddot{I}(t) = o(t), \quad t \rightarrow \infty,$$

implies (3). That it is implied by (3) is more subtle. According to the Lagrange identity [2, p. 235],

$$(6) \quad I = T + h.$$

Since $T \geq 0$, $\ddot{I} \geq h > -\infty$. By a theorem of Landau (see, for example [1, Theorem 1B₂, p. 638]) this property of \ddot{I} entitles us to differentiate each side of (3) to obtain (5). We have proved the equivalence of (2), (3), (5).

Now integrate both sides of (6) and divide by t . Then

$$(7) \quad \frac{I(t)}{t} = \frac{1}{t} \int_0^t T(\tau) d\tau + h + O\left(\frac{1}{t}\right), \quad t \rightarrow \infty.$$

It follows that (5) and (1) are equivalent. Hence so are (1) and (2). The theorem stands established.

The next theorem holds with no a priori assumptions on the growth of S .

THEOREM 2. *We have*

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \int_0^t T(\tau) d\tau \geq |h|.$$

Denote the left-hand side by L . Since $T = -V + h$ and $V \leq 0$ it follows that $T \geq h$. Hence $L \geq h$. On the other hand, according to (7)

$$\limsup_{t \rightarrow \infty} \frac{\dot{I}(t)}{t} = L + h.$$

I claim $L + h \geq 0$; otherwise $\dot{I}(t) \leq -\epsilon t$ for large t , where $\epsilon > 0$. Integrating and dividing by t^2 yields

$$\limsup_{t \rightarrow \infty} \frac{I(t)}{t^2} \leq -\epsilon,$$

which is impossible since $I \geq 0$. Hence $L \geq h$, $L \geq -h$, from which the theorem follows.

COROLLARY. *If $\hat{T} = 0$, then $h = 0$ and $R(t) = o(t)$.*

The first conclusion follows from Theorem 2. Then the second follows from Theorem 1.

I am indebted to Professor M. Golomb for observing that my original proof of the Corollary actually proves the stronger Theorem 2.

REFERENCES

1. R. P. Boas, *Asymptotic relations for derivatives*, Duke Math. J. **3** (1937), 637-646.
2. A. Wintner, *The analytical foundations of celestial mechanics*, Princeton Univ. Press, Princeton, N. J., 1942.

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