TAMING CANTOR SETS IN E^n

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1. **Introduction.** Any two compact, perfect, zero-dimensional and nondegenerate metric spaces are homeomorphic. We call such a space a *Cantor set.* A Cantor set C in Euclidean space E^n is called *tame* if there is a homeomorphism h of E^n onto E^n such that $h(C) \subset E^1 \times \{0_{n-1}\} = E^1 \subset E^n$. For examples of *wild* (i.e., nontame) Cantor sets, see [1], [4], [3], and [9]. The examples of Blankenship [3] give the existence of wild Cantor sets in E^n for each $n \geqq 3$.

Homma [8] and Bing [2, Theorem 5.1] have shown that a Cantor set C in E^3 is tame if and only if $E^3 - C$ is 1-ULC (definition below). It is our purpose here to extend this useful characterization to Cantor sets in E^n ($n \neq 4$). We assume the customary metric on E^n throughout this paper. Let K be a compact set in E^n. Then we say that $E^n - K$ is 1-ULC if for each $\epsilon > 0$ there is a $\delta > 0$ such that each loop (i.e., closed curve) of diameter less than δ in $E^n - K$ is null-homotopic in $E^n - K$ on a set of diameter less than ϵ.

We sketch the proof below, relying heavily on the *cellularity criterion* [10, Theorems 1 and 1′]. For $n \geqq 5$, this criterion implies that a compact absolute retract X in the interior of a piecewise-linear (abbreviated pwl) n-manifold M^n is cellular with respect to piecewise-linear cells if and only if for each open set $U \subset M$ containing X there is an open set V such that $X \subset V \subset U$ and each loop in $V - X$ is null-homotopic in $U - X$.

2. **The theorem.** We first state some lemmas. For Lemma 1, see [11, Theorem 3], [12, Theorem 4], and [6, Theorem 3]. In Whitehead's theorem [12], we take $K = \text{Bd } M$.

LEMMA 1. *Let M^n be a compact piecewise-linear n-manifold* (*possibly with boundary*), *and let E_1 and E_2 be piecewise-linear n-cells in* Int M. *Then there is a piecewise-linear homeomorphism h: $M \to M$ such that $h(E_1) = E_2$ and $h|$*Bd M = *the identity.*

LEMMA 2. *Let C be a Cantor set in E^n, $n \geqq 3$. Then C is tame if for each $\epsilon > 0$ there is a finite, disjoint collection of piecewise-linear n-cells, each of diameter less than ϵ, whose interiors cover C.*

The proof of Lemma 2 is essentially the same as in the three-di-

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mensional case (see [8] and [2, Theorem 1.1]). One uses Lemma 1 to obtain the extensions of homeomorphisms required in the proof.

It follows from [10, Theorem 6] that if $n \neq 4$ and A is a cellular arc in E^n, then each subarc of A is cellular with respect to piecewise-linear cells. Since, for given $\epsilon > 0$, each Cantor set in such an A can be covered by a finite, disjoint collection of arcs each lying in A and each having diameter less than ϵ, Lemma 2 gives the following.

LEMMA 3. *Let A be a cellular arc in E^n, $n \neq 4$. Then each Cantor set in A is tame in E^n.*

THEOREM. *Let C be a Cantor set in E^n, $n \neq 4$. Then C is tame if and only if $E^n - C$ is* 1-ULC.

PROOF. The "only if" half is clear. Suppose now that $E^n - C$ is 1-ULC. By previous remarks, we may assume that $n \geqq 5$. It is not difficult to show that there is an arc A such that $C \subset A \subset E^n$ and A is locally polyhedral except possibly at points of C. By Lemma 3, we have only to verify that the hypotheses of the cellularity criterion are satisfied for A. In fact, we prove the stronger assertion (see [10, Theorem 5]) that $E^n - A$ is 1-ULC.

Toward this end, let $\epsilon > 0$ be given. Choose $\delta > 0$ so that each loop in E^n of diameter less than δ is null-homotopic in E^n on a set of diameter less than ϵ. Now if a pwl loop in $E^n - A$ of diameter less than δ is given, it may be contracted in a pwl manner in E^n on a set of diameter less than ϵ. Since $E^n - C$ is 1-ULC, the contraction may be altered slightly so as to be pwl and to take place in $E^n - C$ (see [5, Theorem 2] and [7, Lemma 2]). Since A is locally a one-dimensional polyhedron away from C and $n \geqq 4$, the contraction may be altered slightly again so as to completely miss A. This completes the proof.

REMARK. If $n = 4$, the proof above still shows that C lies on an arc whose complement is 1-ULC.

COROLLARY 1. *Let Σ^k be a k-sphere, $k \leqq n-1$, topologically embedded in E^n in a locally nice manner, in the sense of* [7], *where $n \geqq 5$. Then each Cantor set in Σ is tame in E^n.*

PROOF. For, [7, Theorem 1] states that each compact absolute retract in Σ is cellular in E^n. Lemma 3 thus may be applied.

COROLLARY 2. *Let C be a Cantor set in E^m and Z a compact* 0-*dimensional set in E^n, where $m, n \geqq 1$. Then the Cantor set $C \times Z$ is tame in $E^m \times E^n = E^{m+n}$.*

PROOF. If $m = n = 1$ or $m = n = 2$, the result follows from the fact that each Cantor set in E^2 is tame. If at least one of m and n is at least two and $m+n\neq 4$, we can appeal to the proof of [**10**, Theorem 7] for the fact that $E^{m+n}-C\times Z$ is 1-ULC and hence, by the present Theorem, $C\times Z$ is tame in E^{m+n}. If (say) $m=3$ and $n=1$, we use Lemma 2 directly. See the first paragraph of the proof of [**10**, Theorem 8] for references.

REMARK. The above Theorem and its corollaries can, of course, be stated and proved with *any* compact 0-dimensional subset of E^n replacing the Cantor set.

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