THE METASTABLE HOMOTOPY OF $O(n)$

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It is not easy to determine how many trivial line bundles can be split off a stable real vector bundle; the first crucial question concerns bundles over a $4k$-sphere. The following result is best possible for the stated spheres:

THEOREM 1. A nontrivial stable real vector bundle over $S^{4k}$ is the sum of an irreducible $(2k+1)$-plane bundle and a trivial bundle, if $k > 4$.

This theorem follows from, and implies, the following theorem. The homotopy group $\pi_q(O(n))$ is stable for $q < n - 1$ (in which case it has been described by Bott [1]), and metastable for $q < 2(n - 1)$. Except for the special cases $n \leq 12$ the metastable groups are related to the stable groups by

THEOREM 2. For $q < 2(n - 1)$ and $n \geq 13$,

$$\pi_q(O(n)) = \pi_q(O) \oplus \pi_{q+1}(V_{2n,n}).$$

In fact, splitting occurs in the homotopy sequence of the fibration $O(2n) \to V_{2n,n}$ at the stated groups. The behaviour in the omitted cases is easily determined from known results.

It follows that the metastable homotopy groups of $O(n)$ exhibit a periodicity, for the second summand is a stable homotopy group of the Stiefel manifold: by [4],

$$\pi_{q+1}(V_{2n,n}) \approx \pi_{q+1}(RP^\infty/RP^{n-1}).$$

Now James has shown [2] that these have a periodicity in a natural way, and in particular that if $t$ denotes the number of nonzero homotopy groups of $O$ in dimensions $\leq q - n$, then

$$\pi_{q+1}(V_{2n,n}) \approx \pi_{q+1+m-n}(V_{2n,m})$$

for all $m \geq n$ such that $m - n$ is divisible by $2^t$. This isomorphism can be induced by a map of the appropriate skeleton of $V_{2n,n}$ into $\Omega^{m-n} V_{2n,m}$, and so is similar to Bott’s periodicity for the stable homotopy groups.

However, the metastable periodicity in $O(n)$ does not arise in exactly the same way as Bott’s. The similarity and the difference are shown by the next theorem.
THEOREM 3. The natural fibration $\Omega^{s*}BSO(n)\to\Omega^{s*}BSO$ has a cross-section over the $(n+4s-7)$-skeleton, but in general $BSO(n)\to BSO$ does not have a cross-section over skeletons of dimension $\geq n$.

It follows that if $g=n+4s-7$, and $t$ (described above) is $\geq 3$, then $\Omega^{s*}BSO(n)$ and $\Omega^{s*+t}BSO(n+2t)$ have the same $q$-type, but $BSO(n)$ and $\Omega^{s*}BSO(n+2t)$ do not have the same $n$-type.

Complete proofs and some applications will appear later; a sketch of the proof of Theorem 1 is given below.

SKETCH PROOF. Theorem 1 is implied by

THEOREM 1*. $\pi_{4k}(BSO(n))\to\pi_{4k}(BSO)$ is trivial if $n\leq 2k$, and onto if $k>4$ and $n\geq 2k+1$.

The first part is easy. For the second part, by Bott periodicity there are homotopy equivalences

$$BSp \cong \Omega^8BSp \cong \Omega^{8m+4}BSO$$

$(m \geq 4)$. so that there are adjoint maps

$$\beta_m: \Sigma^{8m+4}BSp \to BSO, \quad \beta: \Sigma^8BSp \to BSp.$$ 

Then $\beta_m$ includes an epimorphism of homotopy groups in dimensions $\geq 8m+8$, and factorizes into $\beta_{m-1} \circ \Sigma^{8m-4}\beta$ for $m \geq 1$. Calculation of

$$\beta^*: H^{4k}(BSp; Z) \to H^{4k}(\Sigma^8BSp; Z)$$

shows that its image is divisible by 8 if $k$ is odd, and by 4 if $k$ is even.

Now the fibre of $BSO(n)\to BSO(n+4)$ is $V_{n+4,4}$, and the property of $\beta^*$ together with Toda's result [3] that

$$8\pi_{n+r}(V_{n+4,4}) = 0$$

$(n$ odd, $r < n - 1)$, enables classical obstruction theory to prove by induction on $m$, with a little care,

LEMMA 4. $\beta_m: \Sigma^{8m+4}BSp\to BSO$ can be deformed so as to map the $8k$-skeleton into $BSO(8k+1-4m) \subseteq BSO$.

The analogous but more delicate result for the $(8k+8)$-skeleton is too complicated to merit description here. These results are not sharp enough to prove Theorem 1* at once; the proof is concluded by observing that the generator of $\pi_{4k}(BSp)$ can be represented by a composition

$$S^{4k} \xrightarrow{f} X \xrightarrow{g} BSp,$$

where $X$ is a $(4k-16)$-fold suspension of the Cayley plane. The co-
homology maps $f^*, g^*$ can be computed sufficiently accurately for the proof to be completed by the same kind of obstruction argument as before.

REFERENCES