THE CLOSURE OF THE NUMERICAL RANGE CONTAINS THE SPECTRUM[1]

BY EDUARDO H. ZARANTONELLO

Communicated by F. Browder, May 18, 1964

The purpose of this research is to prove that the well-known theorem in the theory of linear operators in Hilbert space [1, p. 147] indicated in the title holds for nonlinear operators and to a certain extent for noncontinuous ones, and to provide a constructive method for solving the equations involved.

DEFINITION 1. The numerical range of a mapping $T: \mathcal{H} \to \mathcal{H}$ of a complex Hilbert space into itself with domain $\mathfrak{D}(T)$ is the set of complex numbers

$$\mathfrak{N}(T) = \left\{ \frac{(Tx_1 - Tx_2, x_1 - x_2)}{\|x_1 - x_2\|^2}, \quad x_1 \neq x_2, \quad x_1, x_2 \in \mathfrak{D}(T) \right\}.$$

In the case of linear mappings, we recall, this is a convex set, which, if the mapping is maximal normal, has a closure coinciding with the convex hull of the spectrum [1, pp. 131, 327].

We shall let $\|T\|$ denote the Lipschitz norm of T, namely,

$$\|T\| = \sup_{x_1 \neq x_2} \frac{\|Tx_1 - Tx_2\|}{\|x_1 - x_2\|}.$$

We shall also use the weaker norm—called the cross-Lipschitz norm —that results from replacing in the above definition the increment $Tx_1 - Tx_2$ by its component orthogonal to $x_1 - x_2$. In general, we define the ν-cross-Hölder norm $(0 \leqq \nu \leqq 1)$ as the quantity

$$\|T\|_\nu^\perp = \sup_{x_1 \neq x_2} \left\{ \left[\|Tx_1 - Tx_2\|^2 - \frac{|(Tx_1 - Tx_2, x_1 - x_2)|^2}{\|x_1 - x_2\|^2} \right]^{1/2} \Big/ \|x_1 - x_2\|^\nu \right\}.$$

If $\|T\|_\nu^\perp < \infty$ we say that T satisfies a cross-Hölder condition of exponent ν. The cross-Lipschitz norm corresponds to $\nu = 1$ and shall simply be denoted $\|T\|^\perp$; for finite-dimensional normal linear mappings it measures the size of the spectrum. In these definitions we have implicitly assumed that the variables range over the whole do-

[1] Research sponsored in part by the National Science Foundation, Grant GP-439.

main of T, but they may be restricted to any subset of $\mathfrak{D}(T)$ to produce relativized notions. Thus a *cross-bounded* mapping T is just one satisfying a 0-cross-Hölder condition over any bounded set, and a locally cross-bounded mapping is a mapping cross-bounded in some neighborhood of each point of its domain.

Continuity will be partially replaced by the following notion where, as usual, whole arrows denote strong convergence, and half arrows weak convergence in $\mathcal{H}$.

DEFINITION 2. A mapping $T: \mathcal{H}\to\mathcal{H}$ is said to be demiclosed if for any directed set of vectors $\{x_\alpha\}$,

$$x_\alpha \to x,\ Tx_\alpha \rightharpoonup y \quad \text{imply} \quad x \in \mathfrak{D}(T) \text{ and } y = Tx.$$

We say that T is sequentially demiclosed if the above is asserted for sequences only.

This closedness is naturally associated with the continuity—often called "demicontinuity"—from the strong topology in the domain to the weak topology in the range.

In the sequel, $\mathcal{B}_r(z)$ will denote the open ball of radius r about z, and $\mathcal{B}_r(z)$ its closure.

These preliminaries settled, we may now state our main result.

THEOREM 1. *Let $T: \mathcal{H}\to\mathcal{H}$ be a locally cross-bounded, sequentially demic osed mapping vanishing at $x=0$, defined on the closed ball $\overline{\mathcal{B}}_r(0)\cap\mathcal{K}$ relatively to a subspace $\mathcal{K}$ dense in $\mathcal{H}$ and demicontinuous over any finite-dimensional convex set therein. Furthermore, let λ be a complex number at a positive distance $d=d(\lambda, T)$ from the numerical range of T. Then, the equation*

$$\lambda x = Tx - y \tag{1}$$

has a unique solution $x\in\overline{\mathcal{B}}_r(0)\cap\mathcal{K}$ for any $y\in\overline{\mathcal{B}}_{rd}(0)$, and the operator $(T-\lambda I)^{-1}$ thus defined is Lipschitzian with norm $\leqq d^{-1}(\lambda, T)$.

The one-to-oneness of the mapping $T_\lambda = T-\lambda I$, and the Lipschitzian character of its inverse result at once from the following inequalities: Since $d(\lambda, T)$ is the distance from λ to the numerical range,

$$\begin{aligned} \left| (T_\lambda x_1 - T_\lambda x_2, x_1 - x_2) \right| &= \left| \frac{(Tx_1 - Tx_2, x_1 - x_2)}{\|x_1 - x_2\|^2} - \lambda \right| \|x_1 - x_2\|^2 \\ &\geqq d(\lambda, T)\|x_1 - x_2\|^2, \end{aligned} \tag{2}$$

for any x_1 and x_2 in $\mathfrak{D}(T)$. Then by Schwarz' inequality,

$$\|T_\lambda x_1 - T_\lambda x_2\| \geqq d(\lambda, T)\|x_1 - x_2\|. \tag{3}$$

Thus the only point in question is the assertion that the range of T_λ covers $\overline{\mathcal{B}}_{rd}(0)$ completely. This is a consequence of the next two lemmas applied to T_λ.

LEMMA 1. *Let $T: \mathcal{H}\to\mathcal{H}$ be a continuous mapping of a finite-dimensional Hilbert space into itself defined on the closed ball $\overline{\mathcal{B}}_r(0)$, vanishing at the origin and satisfying*

$$(4)\qquad |(Tx_1 - Tx_2, x_1 - x_2)| \geqq d\|x_1 - x_2\|^2, \quad x_1, x_2 \in \mathcal{D}(T), \quad d > 0.$$

Then T is one-to-one, its range contains $\overline{\mathcal{B}}_{rd}(0)$, and $\|T^{-1}\| \leqq d^{-1}$.

PROOF. As above, one proves that T is one-to-one with a Lipschitzian inverse. Hence T is a homeomorphism, and by Brouwer's Domain Invariance Theorem [2, p. 156] maps interior points into interior points and sets up a one-to-one correspondence between the boundaries. The origin, being the image of a point interior to the domain, is interior to the range and, since $\|Tx\| \geqq d\|x\|$, is at a distance $\geqq rd$ from the range boundary. Thus $\overline{\mathcal{B}}_{rd}(0) \subset \mathcal{R}(T)$.

LEMMA 2. *Let $T: \mathcal{H}\to\mathcal{H}$ be a cross-bounded mapping, taking the origin into the origin, defined on a closed ball $\overline{\mathcal{B}}_r(0)\cap\mathcal{K}$ relatively to a linear subspace $\mathcal{K}$ dense in $\mathcal{H}$, demicontinuous over any finite-dimensional convex set therein, and satisfying* (4). *Then for any $y\in\overline{\mathcal{B}}_{rd}(0)$ there is a unique $x\in\overline{\mathcal{B}}_r(0)$ and a sequence $\{x_k\}_0^\infty \subset \mathcal{D}(T)$ such that*

$$(5)\qquad x_k \to x, \qquad Tx_k \rightharpoonup y.$$

PROOF. With the help of a sequence $\{\epsilon_k\}_0^\infty$ of positive numbers converging to zero we construct two vector sequences $\{x_k\}_0^\infty$ and $\{z_k\}_0^\infty$ satisfying

$$(6)\qquad x_k \in \overline{\mathcal{B}}_r(0) \cap \mathcal{K}, \qquad \|z_k\| \leqq \epsilon_k, \qquad Tx_k - y + z_k \in \mathcal{K},$$

as follows: We set $x_0 = 0$ and take for z_0 any vector of norm $\leqq \epsilon_0$ such that $Tx_0 - y + z_0 \in \mathcal{K}$. Then, assuming $x_0, \cdots, x_{k-1}, z_0, \cdots, z_{k-1}$ constructed, we take for x_k a vector in the finite-dimensional subspace of $\mathcal{K}$ (letting $\langle\ \rangle$ denote "space spanned by")

$$(7)\qquad \begin{aligned} \mathcal{K}_{k-1} = \langle Tx_0 - y + z_0\rangle \oplus \langle Tx_1 - y + z_1\rangle \oplus \cdots \\ \oplus \langle Tx_{k-1} - y + z_{k-1}\rangle \end{aligned}$$

satisfying the equation

$$(8)\qquad E_{k-1}Tx_k = E_{k-1}y,$$

where E_{k-1} is the orthogonal projection onto $\mathcal{K}_{k-1}$; then we choose for z_k just any vector meeting requirements (6). The restriction of $E_{k-1}T$

to $\overline{\mathcal{B}}_r(0)\cap\mathcal{K}_{k-1}$ is continuous and satisfies (4), and so the existence of x_k is guaranteed by the previous lemma.

Having constructed these sequences we investigate some of their properties. From (8) it follows $E_{k-1}(Tx_k-y)=0$, that is

$$(9)\qquad (Tx_k - y, Tx_h - y + z_h) = 0, \qquad h = 0, 1, \cdots, k-1.$$

Moreover, since $x_0, \cdots, x_k \in \mathcal{K}_{k-1}$,

$$(10)\qquad (Tx_k - y, x_h) = 0, \qquad h = 0, 1, \cdots, k.$$

We are now in a position to show that $Tx_k \rightharpoonup y$. In the first place, since Tx_k-y is orthogonal to x_k-x_0 it is equal to the component of Tx_k-Tx_0 orthogonal to x_k-x_0 minus the corresponding component of y, and hence, by the cross-boundedness of T, its norm is bounded by a constant independent of k. Then, for any u perpendicular to $\bigcup_0^\infty \mathcal{K}_k$,

$$(Tx_k - y, u) = ((Tx_k - y + z_k) - z_k, u) = -(z_k, u)$$

for all values of k, whereas for any $u\in\bigcup_0^\infty \mathcal{K}_k$,

$$(Tx_k - y, u) = 0$$

from a k_0 on. Therefore, $\lim_{k\to\infty}(Tx_k-y, u)=0$ for any u in the dense subspace $(\bigcup_0^\infty \mathcal{K}_k)\oplus(\bigcup_0^\infty \mathcal{K}_k)^\perp$. On account of the boundedness of $\{Tx_k \rightharpoonup y\}_0^\infty$ this implies $Tx_k \rightharpoonup y$.

The x_k's form a bounded sequence and therefore have a weak limit point x at least. Suppose $x_{k_n} \rightharpoonup x$. By (4) and (10), if $k_n > k_m$,

$$(11)\qquad \begin{aligned} d\|x_{k_n} - x_{k_m}\|^2 &\leqq |(Tx_{k_n} - Tx_{k_m}, x_{k_n} - x_{k_m})| \\ &\leqq |((Tx_{k_n} - y) - (Tx_{k_m} - y), x_{k_n} - x_{k_m})| \\ &= |(Tx_{k_m} - y, x_{k_n})|, \end{aligned}$$

whence letting $n\to\infty$ first,

$$(12)\qquad d\|x - x_{k_m}\|^2 \leqq d \limsup_{n\to\infty} \|x_{k_n} - x_{k_m}\|^2 \leqq |(Tx_{k_m} - y, x)|$$

and then $m\to\infty$,

$$(13)\qquad d \limsup_{m\to\infty} \|x - x_{k_m}\|^2 \leqq \limsup_{m\to\infty} |(Tx_{k_m} - y, x)| = 0.$$

Thus we have found an $x\in\overline{\mathcal{B}}_r(0)$ and a sequence $\{x_{k_n}\}$ such that $x_{k_n}\to x$ and $Tx_{k_n}\rightharpoonup y$. The uniqueness of x follows at once from the limiting form of (4) which says that if x' and x'' correspond to y' and y'' respectively, then $|(y'-y'', x'-x'')| \geqq d\|x'-x''\|^2$. This completes the proof.

A number of corollaries can be derived from Theorem 1 by either specializing the hypotheses or by combining them with others. For T's defined on $\overline{\mathfrak{B}}(0)$ the theorem holds under the hypotheses of demicontinuity and cross-boundedness. In this case, however, F. E. Browder—to whom this proof was made available—has been able to prove by a transfinite argument that cross-boundedness is superfluous [3]. Yet, if the mappings are only densely defined this requirement cannot be entirely deleted, as the example of linear symmetric mappings with nonsymmetric adjoints shows [1, p. 149]. The search for weaker conditions to replace cross-boundedness that would perhaps apply to differential operators is a most pressing need of this theory.

An interesting extension brought to the author's attention by G. J. Minty has to do with the idea of replacing the global numerical range—a rather unwieldy object—by the "local closed numerical range"—a smaller and easier to handle set. Letting

$$\mathfrak{N}_r(T) = \left\{\frac{(Tx_1 - Tx_2, x_1 - x_2)}{\|x_1 - x_2\|^2},\ x_1, x_2 \in \mathfrak{D}(T),\quad 0 < \|x_1 - x_2\| \leqq r\right\}, \tag{14}$$

we define the local closed numerical range of T as the set $\mathfrak{N}^*(T) = \bigcap_{r>0} \overline{\mathfrak{N}_r(T)}$. Then, if T is defined all over $\mathfrak{K}$, Theorem 1 remains valid with $\mathfrak{N}^*(T)$ in place of $\mathfrak{N}(T)$.

So far we have only been concerned with the existence of solutions of (1) and their uniqueness, but nothing has been said as to how these solutions could be effectively computed. An adequate technique for this purpose has proved to be that of successive averaging which, under conditions not quite as general as those of Theorem 1, but sufficiently general still, leads step by step to the desired solution, furnishing the theoretical basis for a simple, broad, and flexible computational strategy. The idea is to attain the solution as the limit of successive averages

$$x_k = (1 - \alpha_k)x_{k-1} + \alpha_k\lambda^{-1}(Tx_{k-1} - y) \tag{15}$$

built from an original approximation x_0 by adequate choices of the averaging factors α_k. In expounding this theory we shall assume T defined on a closed ball $\overline{\mathfrak{B}}_r(0)$ and λ will be taken as a complex number $\neq 0$ at a positive distance $d(\lambda, T)$ from the numerical range of T. Due to the fact that T is not everywhere defined, one may not always succeed with schemes like (15), but converging modified averaging schemes like the following

$$x_k = (1 - \alpha_k)x_{k-1} + \alpha_k\lambda^{-1}(Tx_{k-1} - \delta_{k-1}y) \tag{16}$$

can be proved to exist in all cases and, in fact, in very many different ways. This result, however, is still at a purely existential level for so far we know no way of constructing such schemes. It is only when we assume that T satisfies a cross-Hölder condition of exponent $\nu > 1/2$ that a truly recursive procedure for the determinations of the α_k's and δ_k's can be explicitly prescribed. For simplicity's sake we shall confine our attention to cross-Lipschitzian T's. The following two lemmas, which we shall state without proof, form the core of the averaging theory:

LEMMA 3. *If $x_0 \in \overline{\mathcal{B}}_r(0)$ is not a solution of* (1) *and α is such that $x_\alpha = (1-\alpha)x_0 + \alpha\lambda^{-1}(Tx_0 - y) \in \overline{\mathcal{B}}_r(0)$, then*

$$\frac{\|T_\lambda x_0 - y\|^2 - \|T_\lambda x_\alpha - y\|^2}{\|T_\lambda x_0 - y\|^2} \geqq R(1 - |\gamma|^2), \tag{17}$$

where

$$R = \frac{d^2(\lambda, T)}{d^2(\lambda, T) + (\|T\|_\perp)^2}, \tag{18}$$

$$\gamma = \frac{\alpha}{R\lambda} \frac{(T_\lambda x_0 - T_\lambda x_\alpha, x_0 - x_\alpha)}{\|x_0 - x_\alpha\|^2} + 1. \tag{19}$$

This lemma points to the fundamental fact that the error committed in solving equation (1) could be made strictly smaller from one approximation to the next if one only knew that α can be chosen so that $|\gamma| < 1$. This calls for a study of the mapping $\alpha \to \gamma$, which is precisely the content of the next lemma.

LEMMA 4. *If $x_0 \in \mathcal{B}_r(0)$, and $\|y\| + \|T_\lambda x_0 - y\| \leqq rd(\lambda, T)$, then the mapping of the complex plane into itself*

$$\alpha \to \gamma = \frac{\alpha}{R\lambda} \frac{(T_\lambda x_0 - T_\lambda x_\alpha, x_0 - x_\alpha)}{\|x_0 - x_\alpha\|^2} + 1 \tag{20}$$

is a homeomorphism having as domain the set of α's for which $\|x_\alpha\| \leqq r$ (closed circular disc), and as range a closed domain containing the closed unit disc about the origin. Furthermore, its inverse is Lipschitzian with norm $\leqq R|\lambda|/d(\lambda, T)$.

We have thus obtained more than we had the right to hope for, because it is not only possible to find an α for which $|\gamma| < 1$, but for any such γ an α exists. The combination of these two lemmas leads, after some manipulation, to the final result:

THEOREM 2. *Let $T = \mathcal{H} \to \mathcal{H}$ be a continuous, cross-Lipschitzian mapping defined on $\overline{\mathcal{B}}_r(0)$ vanishing at the origin, λ a complex number* *different from zero at a positive distance* $d(\lambda, T)$ *from the numerical range of* T, *and* y *a vector in* $\overline{\mathcal{B}}_{rd}(0)$. *Then for any sequence of complex numbers* $\{\gamma_k\}_1^\infty$ *satisfying*

$$(21)\qquad |\gamma_k| \leqq 1, \qquad \sum_1^\infty (1 - |\gamma_k|^2) = \infty,$$

the vector sequence $\{x_k\}_0^\infty$ *starting with* $x_0 = 0$ *and recursively constructed as follows*:

$$(22)\qquad x_k = (1 - \alpha_k)x_{k-1} + \alpha_k\lambda^{-1}(Tx_{k-1} - \delta_{k-1}y),$$

where

$$(23)\qquad \delta_{k-1} = 1 - \left(\prod_{h=1}^{k-1} \frac{1}{2}\{1 + [1 - R(1 - |\gamma_n|^2)]^{1/2}\}\right)(1 - \delta_0), \qquad 0 < \delta_0 \leqq \frac{1}{2},$$

and α_k *satisfies*

$$(24)\qquad \|x_k - x_{k-1}\|^2(\gamma_k - 1) = \frac{\alpha_k}{R\lambda}(T_\lambda x_k - T_\lambda x_{k-1}, x_k - x_{k-1}),$$

can be continued indefinitely and converges to the solution of (1). *Further, if* x *is the solution,*

$$(25)\qquad \|x_k - x\| \leqq \frac{2(1 - \delta_0)\|y\|}{d(\lambda, T)} \prod_{n=1}^{k} \frac{1}{2}\{1 + [1 - R(1 - |\gamma_n|^2)]^{1/2}\}.$$

The theory further asserts that if the numerical range is either bounded, or viewed from λ under an angle less than π, or both, then there are choices of the γ_k's satisfying (21) for which the corresponding α_k's are all of the same absolute values in the first case, of the same argument in the second, and are all equal in the third, these assertions holding asymptotically if the corresponding properties hold only locally. Moreover, if $\|y\| \leqq rd(\lambda, T)/2$ then the δ_k's can all be replaced by 1.

Bibliography

1. M. H. Stone, *Linear transformations in Hilbert space*, Amer. Math. Soc. Colloq. Publ. Vol. 15, Amer. Math. Soc., Providence, R. I., 1932.

2. P. A. Alexandroff, *Combinatorial topology*, Graylock, Rochester, N. Y., 1956.

3. F. E. Browder, *Remarks on nonlinear functional equations*, Proc. Nat. Acad. Sci. U.S.A. **51** (1964), 985–989.

Universidad Nacional de Cordoba, Argentina
University of Kansas