EXTENSIONS OF HAAR MEASURE FOR COMPACT CONNECTED ABELIAN GROUPS

BY GERALD L. ITZKOWITZ[1]

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We outline in this paper generalizations of some theorems of Hulanicki on the existence of dense subsets of small cardinality in product measure spaces and in compact groups. We then apply a special case of these results to show the existence of the Kakutani-Oxtoby measure for the case of compact connected Abelian topological groups. A more detailed paper will appear later on.

DEFINITION. Let $\mathfrak{A}$, $\mathfrak{B}$ be collections of nonvoid sets of a space X. Then $\mathfrak{A}$ is a weak base for $\mathfrak{B}$ if and only if given $B\in\mathfrak{B}$ there is an $A\in\mathfrak{A}$ such that $A\subset B$.

If A is a set then $|A|$ denotes the cardinal of A; $\mathfrak{n}$ will always denote an infinite cardinal.

The following theorem generalizes Hulanicki [7, Theorem 1].

THEOREM 1. *Let $X=P_{t\in T}X_t$, where $\{(X_t, \mathfrak{B}_t): t\in T\}$ is a family of measurable spaces, each having a weak base of cardinal at most $\mathfrak{n}^{\aleph_0}$, and $|T|\leqq 2^{\mathfrak{n}}$. Then the product measurable space $(X, \mathfrak{B})$ has a weak base $\mathfrak{A}$ for the σ-field $\mathfrak{B}$ for which $|\mathfrak{A}|\leqq\mathfrak{n}^{\aleph_0}$.*

The proof uses the following lemma.

LEMMA 1. *Let T be any set such that $|T|=2^{\mathfrak{n}}$; then there exists a family $\mathfrak{A}$ of sequences $\{B_i\}_{i=1}^{\infty}$ of pairwise disjoint subsets of T such that*

(i) $|\mathfrak{A}|\leqq\mathfrak{n}^{\aleph_0}$,

(ii) *for any distinct sequence $\{t_i\}_{i=1}^{\infty}$ in T, there exists a sequence $\{B_i\}_{i=1}^{\infty}\in\mathfrak{A}$ such that $t_i\in B_i$ for each i.*

This lemma can be proved by noticing that there is a 1-1 correspondence of T with $\{-1, 1\}^{\mathfrak{n}}$, and this latter set has at most $\mathfrak{n}^{\aleph_0}$ closed G_δ sets.

Let X be a topological space. Let $w(X)$ denote the least cardinal of a basis of open sets for X. It is not difficult to show that if H is a compact Abelian group and if $w(H)\leqq\mathfrak{n}$, then H has at most $\mathfrak{n}^{\aleph_0}$ closed G_δ sets. Thus, trivially, there is a weak base for the Baire sets of H having cardinal at most $\mathfrak{n}^{\aleph_0}$.

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COROLLARY 1. *Let $G = P_{t\in T} H_t$, where each $H_t = H$, H is a compact Abelian group, $w(H) \leqq \mathfrak{n}$, and $|T| \leqq 2^{\mathfrak{n}}$. Then $w(G) \leqq 2^{\mathfrak{n}}$ and there is a weak base for the Baire sets of G having cardinal at most $\mathfrak{n}^{\aleph_0}$.*

Kakutani [8] has shown that if $w(H) = \mathfrak{n}$, then $|H| = 2^{\mathfrak{n}}$ (see also [3, 24.47]). Thus the following holds.

COROLLARY 2. *Let $G = P_{t\in T} H_t$, where each $H_t = H$, H is a compact Abelian group, $w(H) \leqq \mathfrak{n}$, and $|T| \leqq 2^{2^{\mathfrak{n}}}$. Then there is a weak base for the Baire sets of G having cardinal at most $2^{\mathfrak{n}}$.*

COROLLARY 3. *Let G be as in Corollary 2. Then G has a dense pseudocompact subgroup J which necessarily has Haar outer measure one and $|J| \leqq 2^{\mathfrak{n}}$.*

This follows from a theorem of Comfort and Ross [1] which states that a totally bounded group G is pseudocompact if and only if each nonempty Baire subset of $\overline{G}$ meets G, where $\overline{G}$ is the Weil completion of G. We note in the proof that if H is a compact group and if $A \subset H$, then A has Haar outer measure one if and only if $A \cap B \neq \emptyset$ for each Baire set B of positive measure.

THEOREM 2. *Let G be a compact Abelian topological group satisfying $w(G) = 2^{\mathfrak{n}}$ for some infinite cardinal number $\mathfrak{n}$. Then*

(i) *G has a weak base for its Baire sets of cardinal at most $\mathfrak{n}^{\aleph_0}$,*

(ii) *G contains a dense pseudocompact subgroup J such that $|J| \leqq \mathfrak{n}^{\aleph_0}$; necessarily J has outer measure one.*

This theorem is proved by using Corollary 1 and the following theorem of Vilenkin [11]: Let G be a compact Abelian group. For some cardinal number $\mathfrak{m}$, there is a continuous mapping of $\{-1, 1\}^{\mathfrak{m}}$ onto G; $\mathfrak{m}$ can be taken to be $\max[\aleph_0, r]$, where r is the rank of the character group of G.

We may observe that Theorem 2 is a generalization of a theorem of Hartman and Hulanicki [2]: If G is a compact group satisfying $|G| \leqq 2^{2^{\mathfrak{n}}}$ and if the generalized continuum hypothesis holds, then there is a dense subgroup $H \subset G$ satisfying $|H| \leqq \mathfrak{n}$. We note here that we did not use the generalized continuum hypothesis. Finally, part (i) of Theorem 2 appears to contain Theorem 2 of Hulanicki [7].

We next prove a special case of Corollary 2. We note that Corollary 2 is an existence theorem. We will now construct a set that is actually a weak base for the closed G_δ sets of G in Corollary 2.

Let G be as in Corollary 2. Let $\mathfrak{N}$ be the collection of closed G_δ sets of H. As above we note that $|\mathfrak{N}| \leqq \mathfrak{n}^{\aleph_0}$. Let $\mathfrak{A}$ be the collection of sequences of pairwise disjoint sets in T satisfying (i) and (ii) of Lemma 1.

DEFINITION. An $(\mathfrak{A}, \mathfrak{N})$-cylinder set in G is a set of the form $M=\bigcap_{i=1}^{\infty}\{\bigcap_{t(i)\in B(i)}\pi_{t(i)}^{-1}(N_{t(i)})\}$, where $\{B(i)\}_{i=1}^{\infty}\in\mathfrak{A}$, and for each i, all $N_{t(i)}=N_i$ for some $N_i\in\mathfrak{N}$.

Let $\mathfrak{C}_\delta$ be the collection of all $(\mathfrak{A}, \mathfrak{N})$-cylinder sets in G. It is immediate from Lemma 1 that $|\mathfrak{C}_\delta|\leqq 2^{\mathfrak{n}}$.

THEOREM 3. *Let G be as in Corollary 2. Then $\mathfrak{C}_\delta$ is a weak base for the closed G_δ sets in G.*

The proof of this theorem uses the following lemma and the reflexivity of the property of being a weak base.

LEMMA 2. *Let G be as in Corollary 2. Then the collection $\mathcal{G}_\delta$ of all nonvoid closed G_δ sets in G of the form $\bigcap_{i=1}^{\infty}\pi_{t(i)}^{-1}(N_{t(i)})$, where $\{t(i)\}_{i=1}^{\infty}\subset T$, and $N_{t(i)}\in\mathfrak{N}$ for each i, is a weak base for the closed G_δ sets in G.*

Kakutani and Oxtoby [10] proved that Haar measure in a compact metric group may be extended to a much larger σ-field of subsets of the group and still remain invariant under group translation and inversion. To be more precise we introduce the following definition.

DEFINITION. The character of a measure space $(X, \mathcal{S}, \mu)$ is the smallest cardinal number $\mathfrak{m}$ for which there is a subfamily $\mathcal{R}\subset\mathcal{S}$ such that $|\mathcal{R}|=\mathfrak{m}$ and such that for each $S\in\mathcal{S}$ and each $\epsilon>0$, there exists a set $R\in\mathcal{R}$ satisfying $\mu(S\triangle R)<\epsilon$.

It is well known that the character of the Haar measure space of a compact infinite metric group is $\aleph_0$. Kakutani and Oxtoby showed that there is an extension of Haar measure with character $2^{\mathfrak{c}}$.

Kakutani and Kodaira [9] showed that there is an extension of Haar measure on the circle of character $\mathfrak{c}$. Hulanicki [7], using Theorem 1 of his paper, showed that the method of Kakutani and Kodaira may be used to get an extension of character $2^{\mathfrak{c}}$.

THEOREM 4. *Let H be a compact connected Abelian topological group satisfying $w(H)=\mathfrak{n}$. Then there exists a translation- and inversion-invariant extension of Haar measure on H of character $2^{2^{\mathfrak{n}}}$.*

We remark that for a compact infinite Abelian group G it is easy to show that the character of the Haar measure space of G is equal to $w(G)$. Thus the character of the Haar measure space of H in the above theorem is $\mathfrak{n}$. Our method of proof of Theorem 4 is similar to that of Kakutani and Kodaira. We briefly outline the proof in the following theorem and lemmas.

THEOREM 5. *Let G be as in Corollary 2. Let $\beta\in T$ be fixed. Let $\mathcal{P}_\beta\subset\mathfrak{C}_\delta$ consist of those $(\mathfrak{A}, \mathfrak{N})$-cylinder sets of the form*

$\bigcap_{n=1}^{\infty}\{\bigcap_{t(n)\in B(n)}\pi_{t(n)}^{-1}(N_n)\}$ *that satisfy* $\beta\in B(i)$ *for some* i *and* N_i *has positive Haar measure in* H *for this* i. *Then* $\mathcal{P}_\beta$ *is a weak base for the closed* G_δ *sets in* G *having positive Haar measure.*

REMARK. It is clear from the construction of $\mathcal{P}_\beta$ that $|\mathcal{P}_\beta|\leqq 2^{\mathfrak{n}}$ and if $A\in\mathcal{P}_\beta$ then $\pi_\beta(A)$ has positive Haar measure in H_β and is a closed G_δ there (π_β is the projection onto H_β).

LEMMA 3. *Let* G *be a compact Abelian topological group. Let* $M\subset G$ *be a set of positive Haar measure. Then* M *contains a maximal independent set of elements of infinite order in* G.

This lemma is a consequence of a well-known theorem which states (using additive notation) that if M has positive Haar measure in G then $M-M$ contains the identity in its interior. It follows then that the group $[M]$ generated by M has finite index in G if G is compact and hence every element of infinite order in G is dependent on M.

LEMMA 4. *Let* G *be a compact connected Abelian topological group satisfying* $w(G)=\mathfrak{n}$. *Then every closed* G_δ *set* $M\subset G$ *having positive Haar measure contains a maximal linearly independent set* L *of elements of infinite order in* G *and* $|L|=2^{\mathfrak{n}}$.

This lemma follows from Lemma 3, the fact that all maximal linearly independent sets of elements of infinite order have the same cardinality, and a structure theorem of Hulanicki [5], [6] for compact connected Abelian groups. Lemma 4 allows us to carry out a transfinite induction which leads to:

LEMMA 5. *Let* G *be a compact connected Abelian group satisfying* $w(G)=\mathfrak{n}\geqq\aleph_0$. *Let* $\{M_\alpha:\alpha<\omega_{\mathfrak{m}},\ \mathfrak{m}=2^{\mathfrak{n}}\}$ *be a well-ordered sequence of closed* G_δ *sets of positive Haar measure in* G. *Then there exists a well-ordered set* $\{x_\alpha:\alpha<\omega_{\mathfrak{m}}\}$ *of independent elements of infinite order such that* $x_\alpha\in M_\alpha$ *for each* $\alpha<\omega_{\mathfrak{m}}$. (*The* M_α*'s are not necessarily distinct.*)

REMARK. Lemma 5 is true in a more general situation. The same induction will work because of Lemma 3 if the M_α are measurable with positive measure, $\mathfrak{m}$ is at most equal to the cardinal of a maximal independent set of elements of infinite order, and G is compact Abelian (with no other restrictions).

LEMMA 6. *Let* H *be a compact connected Abelian group satisfying* $w(H)=\mathfrak{n}\geqq\aleph_0$. *Let* $G=\boldsymbol{P}_{t\in T}H_t$ *where each* $H_t=H$ *and* $|T|=2^{2^{\mathfrak{n}}}$. *Fix the coordinate* $\beta\in T$. *Then there is a set* $V\subset G$ *of independent elements of infinite order satisfying*

(i) *V has Haar outer measure one,*

(ii) $\pi_\beta|_V$ *is one-to-one.*

This is proved by using Theorem 5 (i.e., projecting onto H_β the elements of $\mathcal{P}_\beta$) and then using Lemma 5.

Letting V_G be the free group generated by V, and letting W be the free group generated by $\pi_\beta(V)$, it is easy to see that π_β induces an algebraic isomorphism ϕ of W onto V_G. Furthermore, ϕ may be extended to an algebraic isomorphism of H into G satisfying $\pi_\beta\phi(x)=x$ for all $x\in H_\beta$, because H and G are divisible. It follows that $\phi(H_\beta)$ is a group of outer measure one in G. Thus the remainder of the proof of Theorem 4 is a repetition of the final part of the proof of Kakutani and Kodaira [9] for the circle.

REMARK. One could use the method of proof outlined above without Theorem 5 to show the existence of an extension of Haar measure of character $2^{\mathfrak{n}}$.

NOTE. Since this work was completed, Hewitt and Ross [4], have generalized and simplified Theorem 4; their theorem implies Theorem 4 for all compact Abelian groups, and uses our Theorem 2, Lemma 3, and Lemma 5 with the remark following it.

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UNIVERSITY OF ROCHESTER