AN EIGENVALUE PROBLEM FOR QUASI-LINEAR ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS

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Eigenvalue problems for nonlinear equations have long been studied in the contexts of abstract function spaces and second-order ordinary differential equations. The present note treats such problems for certain quasi-linear elliptic partial differential equations by means of functional analysis on Sobolev spaces, and extends work in this direction by Levinson [7], Golomb [6], Duff [5], and Vaĭnberg [8]. The variational method used is a direct generalization of the linear case and thus allows the introduction of a simple Hilbert-space approach to this problem.

1. Let G be a fixed bounded domain in real Euclidean N-space R^N with boundary G and closure $\bar{G}=G\cup\partial G$. A general point of G will be denoted $x=(x_1, x_2, \cdots, x_n)$. Integration over G will always be taken with respect to Lebesgue N-dimensional measure. All derivatives are taken in the generalized sense of L. Schwartz. The following notation is very convenient: the elementary differential operators are written

$$D_j = \frac{1}{i}\frac{\partial}{\partial x_j} \qquad (1 \leqq j \leqq N),$$

and for any N-tuple of non-negative integers $\alpha=(\alpha_1, \alpha_2, \cdots, \alpha_N)$ the corresponding differential operator of order $|\alpha| = \alpha_1+\alpha_2+ \cdots +\alpha_N$ is written $D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} \cdots D_N^{\alpha_N}$. A linear operator A of order $2m$ is said to be in divergence form if it can be written:

$$Au = \sum_{|\alpha|,|\beta|\leqq m} D^\alpha(a_{\alpha\beta}(x)D^\beta u).$$

If $a_{\alpha\beta}(x)=a_{\beta\alpha}(x)$, A is also formally self-adjoint.

A real linear differential operator A is uniformly elliptic in G if the

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homogeneous characteristic form of A is positive definite, uniformly over G.

$W_{m,p}(G)$ is the collection of functions in $L_p(G)$ for fixed p, $1 < p < \infty$, such that $D^\alpha u$, for all $|\alpha| \leqq m$, again lies in $L_p(G)$. $W_{m,p}(G)$ is a Banach space with respect to the norm

$$\|u\|_{m,p} = \left\{ \sum_{|\alpha| \leqq m} \|D^\alpha u\|_{0,p}^p \right\}^{1/p}.$$

In particular, $W_{m,2}(G)$ is a Hilbert space with respect to the inner product

$$\langle u, v \rangle_{m,2} = \sum_{|\alpha| \leqq m} \langle D^\alpha u, D^\alpha v \rangle_{0,2}.$$

$\mathring{W}_{m,2}(G)$ is the closure of $C_0^\infty(G)$ in $W_{m,2}(G)$ and thus can be regarded as a Hilbert space.

2. The boundary-value problem to be considered here is:

$$Au - \lambda f(u, x) = 0, \qquad D^\alpha u\,|_{\partial G} = 0, \quad 0 \leqq |\alpha| \leqq m - 1, \tag{1}$$

where A is a formally self-adjoint, uniformly elliptic real linear operator of order $2m$ with uniformly bounded, measurable coefficients and top order terms uniformly continuous (A is assumed to be written in divergence form), λ is a real number and $f(t, x)$ is a real-valued function defined on $R^1 \times G$, jointly continuous in the x and t variables with the following properties:

1. $f(0, x) = 0$.
2. $f(-t, x) = -f(t, x)$.
3. $f(t, x)$ is a nondecreasing function of t for fixed x.
4. For some fixed $x_1 \in G$, some positive constant k and all $x \in G$, $f(t, x) \geqq kf(t, x_1) > 0$, for $t > 0$.
5. (*Polynomial growth condition*) For all $(t, x) \in R^1 \times G$, $f(t, x) \leqq k_1 + k_2 |t|^\rho$, where k_1 and k_2 are non-negative constants and $\rho = \rho(m, N)$. If $f(t, x)$ does not (necessarily) satisfy this condition, we write $\rho = \infty$.

We denote by $Z(\rho_1)$ the family of functions $f(t, x)$ which satisfy the above conditions with

$$0 \leqq \rho < \frac{N + 2m}{N - 2m} \quad \text{if } N > 2m,$$

$$0 \leqq \rho < \infty \quad \text{if } N = 2m,$$

$$\rho = \infty \quad \text{if } N < 2m.$$

DEFINITION 1. A *classical eigenfunction* for the boundary-value problem (1) is a function $u(x)$ with the following properties:

(a) $u(x)$ is $2m$-times continuously differentiable over G.

(b) $u(x)$ is $(m-1)$-times continuously differentiable over $\overline{G}$.

(c) $u(x)$ satisfies the equation $Au=\lambda f(u, x)$ in G.

(d) $u(x)\not\equiv 0$ in G.

(e) $D^\alpha u\big|_{\partial G}=0$, $0\leqq|\alpha|\leqq m-1$, at each point x of ∂G.

DEFINITION 2. A *generalized eigenfunction* of the boundary-value problem (1) is a function $u(x)$ with the following properties:

(a) $u(x)\in \mathcal{W}_{m,2}(G)$.

(b) $u(x)\not\equiv 0$, apart from a set of measure 0, in G.

(c) $a(u, v)=\lambda\int_G f(u, x)v$ for all $v\in\mathcal{W}_{m,2}(G)$, where $a(u, v)$ is the Dirichlet form associated with the operator A, i.e.,

$$a(u,v) = \sum_{|\alpha|,|\beta|\leq m} \int_G a_{\alpha\beta}(x) D^\alpha u D^\beta v.$$

Differentiation by parts shows that every classical eigenfunction is a generalized eigenfunction. The converse is, in general, not true. (Cf. Theorem III.)

3. THEOREM I. *Suppose* $f(t, x)\in Z(\rho_1)$. *Then the generalized eigenfunctions of the boundary-value problem* (1) *are identical with the nonzero solutions of the operator equation* $\mathfrak{A}u-\lambda Bu=0$ *defined on the Hilbert space* $\mathcal{W}_{m,2}(G)$; $\mathfrak{A}$ *is a self-adjoint bounded linear operator mapping* $\mathcal{W}_{m,2}(G)$ *into itself and satisfying the inequality*

$$\langle \mathfrak{A}u, u\rangle_{m,2} \geqq c_1\|u\|^2_{m,2} - c_2\|u\|^2_{0,2},$$

where c_1, c_2 *are constants with* $c_1>0$ *and* $c_2\geqq 0$; B *is a compact, continuous, not necessarily linear mapping of* $\mathcal{W}_{m,2}(G)$ *into itself with the additional property that* $\langle Bu, v\rangle_{m,2}$ *is a weakly continuous function of the elements u and v.*

PROOF. Define $\langle \mathfrak{A}u, v\rangle_{m,2}=a(u, v)$ and $\langle Bu, v\rangle_{m,2}=\int_G f(u, x)v$. Noticing that both inner products are linear in v, we are able to apply Riesz's representation theorem for linear functionals on the Hilbert space $\mathcal{W}_{m,2}(G)$ to obtain the required operator equation. The inequality satisfied by $\mathfrak{A}$ is a consequence of Gårding's Inequality, while the properties of B follow from Sobolev's Imbedding Theorem.

Set $F(t, x)=\int_0^t f(s, x)ds$.

DEFINITION 3. ∂M_R is the set of all functions $u(x)$ such that:

(a) $u(x)\in\mathcal{W}_{m,2}(G)$,

(b) $\int_G F(u, x)=R$.

(We refer to ∂M_R as the energy level with radius R.)

LEMMA 1 (GEOMETRY OF ENERGY LEVELS). *Let R be a fixed positive number. Then*

(i) ∂M_R *contains elements of* $\mathcal{W}_{m,2}(G)$.

(ii) ∂M_R *is weakly closed and, on* ∂M_R, $\|u\|_{m,2} \geq c(R) > 0$, *for some constant* $c(R)$ *independent of u.*

(iii) *On* ∂M_R, $\|u\|_{0,1} \leq g(R)$, *where* $g(R)$ *is a constant independent of u.*

THEOREM II (EXISTENCE THEOREM). *Let G be any bounded domain* R^N. *Suppose* $f(t, x) \in Z(\rho_1)$. *Then the boundary-value problem* (1) *has a generalized eigenfunction* $u(x)$; $u(x)$ *is normalized by the requirement that* $u(x) \in \partial M_R$ *for some fixed positive R and characterized as a solution of the variational problem* inf $a(v, v)$ *for* $v \in \partial M_R$.

This result is proved by solving the above variational problem by the direct method of the calculus of variations, using the geometrical properties of ∂M_R, as illustrated in Lemma 1. We then show that a solution of the variational problem is also a solution of the operator equation.

We note the following regularity conditions associated with the boundary-value problem (1): (1a) G is of class $4m$. (1b) For the coefficients of A, $a_{\alpha\beta}(x) \in C^{2m}(\overline{G})$, (1c) $f(t, x)$ satisfies a local Lipschitz condition in t and a local Hölder condition of exponent r, $0 < r < 1$, for $x \in G$.

THEOREM III (REGULARITY THEOREM). *Suppose the regularity conditions* (1a), (1b), (1c) *for the boundary-value problem* (1) *hold, and* $f(t, x) \in Z(\rho_1)$. *Then any generalized eigenfunction for* (1) *is a classical eigenfunction.*

This result follows immediately from the results of Agmon [**1**], Agmon, Douglis and Nirenberg [**2**] and Browder [**3**], [**4**] on the L_p regularity theory of elliptic operators.

THEOREM IV (POSITIVE EIGENFUNCTIONS). *Let the hypotheses of Theorem* II *be satisfied. Then if A is a second-order operator, the boundary-value problem* (1) *has a positive generalized eigenfunction* $u(x)$ *in G. If* $u(x)$ *is a classical eigenfunction, then* $u(x) > 0$ *in G.*

REMARK. The boundary-value problem studied here provides another example of nonuniqueness in the theory of quasi-linear elliptic equations of the type similar to the Navier-Stokes equation for a stationary flow of an incompressible fluid.

Added in proof (*December* 7, 1964). A variational method can also be used to prove the existence of an infinite number of distinct normalized eigenfunctions u_n with associated eigenvalues λ_n, tending to infinity with n, for the boundary value problem (I).

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