AN EIGENVALUE PROBLEM FOR QUASI-LINEAR ELLIPTIC
PARTIAL DIFFERENTIAL EQUATIONS

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Eigenvalue problems for nonlinear equations have long been
studied in the contexts of abstract function spaces and second-order
ordinary differential equations. The present note treats such problems
for certain quasi-linear elliptic partial differential equations by means
of functional analysis on Sobolev spaces, and extends work in this
direction by Levinson [7], Golomb [6], Duff [5], and Valnberg [8].
The variational method used is a direct generalization of the linear
case and thus allows the introduction of a simple Hilbert-space ap­
proach to this problem.

1. Let $G$ be a fixed bounded domain in real Euclidean $N$-space $\mathbb{R}^N$
with boundary $G$ and closure $\overline{G} = G \cup \partial G$. A general point of $G$
will be denoted $x = (x_1, x_2, \cdots, x_n)$. Integration over $G$
will always be
taken with respect to Lebesgue $N$-dimensional measure. All deriva­
tives are taken in the generalized sense of L. Schwartz. The following
notation is very convenient: the elementary differential operators are
written

$$D_j = \frac{1}{i} \frac{\partial}{\partial x_j} \quad (1 \leq j \leq N),$$

and for any $N$-tuple of non-negative integers $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_N)$
the corresponding differential operator of order $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_N$
is written $D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} \cdots D_N^{\alpha_N}$. A linear operator $A$
of order $2m$
is said to be in divergence form if it can be written:

$$Au = \sum_{|\alpha|, |\beta| \leq m} D^\alpha(a_{\alpha\beta}(x) D^\beta u).$$

If $a_{\alpha\beta}(x) = a_{\beta\alpha}(x)$, $A$ is also formally self-adjoint.

A real linear differential operator $A$ is uniformly elliptic in $G$ if the

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homogeneous characteristic form of $A$ is positive definite, uniformly over $G$.

$W_{m,p}(G)$ is the collection of functions in $L_p(G)$ for fixed $p$, $1 < p < \infty$, such that $D^\alpha u$, for all $|\alpha| \leq m$, again lies in $L_p(G)$. $W_{m,p}(G)$ is a Banach space with respect to the norm

$$
\|u\|_{m,p} = \left\{ \sum_{|\alpha| \leq m} \|D^\alpha u\|_{0,p}^p \right\}^{1/p}.
$$

In particular, $W_{m,2}(G)$ is a Hilbert space with respect to the inner product

$$
\langle u, v \rangle_{m,2} = \sum_{|\alpha| \leq m} \langle D^\alpha u, D^\alpha v \rangle_{0,2}.
$$

$W_{m,2}(G)$ is the closure of $C_0^\infty(G)$ in $W_{m,2}(G)$ and thus can be regarded as a Hilbert space.

2. The boundary-value problem to be considered here is:

$$
Au - \lambda f(u, x) = 0,
$$

$$
D^\alpha u |_{\partial G} = 0, \quad 0 \leq |\alpha| \leq m - 1,
$$

where $A$ is a formally self-adjoint, uniformly elliptic real linear operator of order $2m$ with uniformly bounded, measurable coefficients and top order terms uniformly continuous ($A$ is assumed to be written in divergence form), $\lambda$ is a real number and $f(t, x)$ is a real-valued function defined on $\mathbb{R}^1 \times G$, jointly continuous in the $x$ and $t$ variables with the following properties:

1. $f(0, x) = 0$.
2. $f(-t, x) = -f(t, x)$.
3. $f(t, x)$ is a nondecreasing function of $t$ for fixed $x$.
4. For some fixed $x_1 \in G$, some positive constant $k$ and all $x \in G$, $f(t, x) \geq kf(t, x_1) > 0$, for $t > 0$.
5. (Polynomial growth condition) For all $(t, x) \in \mathbb{R}^1 \times G$, $f(t, x) \leq k_1 + k_2 \left| t \right|^\rho$, where $k_1$ and $k_2$ are non-negative constants and $\rho = \rho(m, N)$. If $f(t, x)$ does not (necessarily) satisfy this condition, we write $\rho = \infty$.

We denote by $Z(\rho)$ the family of functions $f(t, x)$ which satisfy the above conditions with

$$
0 \leq \rho < \frac{N + 2m}{N - 2m} \quad \text{if } N > 2m,
$$

$$
0 \leq \rho < \infty \quad \text{if } N = 2m,
$$

$$
\rho = \infty \quad \text{if } N < 2m.
$$
DEFINITION 1. A classical eigenfunction for the boundary-value problem (1) is a function \( u(x) \) with the following properties:

(a) \( u(x) \) is \( 2m \)-times continuously differentiable over \( G \).
(b) \( u(x) \) is \( (m-1) \)-times continuously differentiable over \( \overline{G} \).
(c) \( u(x) \) satisfies the equation \( Au = \lambda f(u, x) \) in \( G \).
(d) \( D^\alpha u \big|_{\partial G} = 0 \), \( 0 \leq |\alpha| \leq m-1 \), at each point \( x \) of \( \partial G \).

DEFINITION 2. A generalized eigenfunction of the boundary-value problem (1) is a function \( u(x) \) with the following properties:

(a) \( u(x) \in W_{m,2}(G) \).
(b) \( u(x) \neq 0 \), apart from a set of measure 0, in \( G \).
(c) \( a(u, v) = \lambda \int_G f(u, x)v \) for all \( v \in W_{m,2}(G) \), where \( a(u, v) \) is the Dirichlet form associated with the operator \( A \), i.e.,

\[
a(u, v) = \sum_{|\alpha|,|\beta|\leq m} \int_G a_{\alpha\beta}(x) D^\alpha u D^\beta v.
\]

Differentiation by parts shows that every classical eigenfunction is a generalized eigenfunction. The converse is, in general, not true. (Cf. Theorem III.)

3. THEOREM I. Suppose \( f(t, x) \in L(p_1) \). Then the generalized eigenfunctions of the boundary-value problem (1) are identical with the non-zero solutions of the operator equation \( A u - \lambda Bu = 0 \) defined on the Hilbert space \( W_{m,2}(G) \); \( A \) is a self-adjoint bounded linear operator mapping \( W_{m,2}(G) \) into itself and satisfying the inequality

\[
\langle A u, u \rangle_{m,2} \geq c_1 \| u \|^2_{m,2} - c_2 \| u \|^2_{0,2},
\]

where \( c_1, c_2 \) are constants with \( c_1 > 0 \) and \( c_2 \geq 0 \); \( B \) is a compact, continuous, not necessarily linear mapping of \( W_{m,2}(G) \) into itself with the additional property that \( \langle Bu, v \rangle_{m,2} \) is a weakly continuous function of the elements \( u \) and \( v \).

PROOF. Define \( \langle u, v \rangle_{m,2} = a(u, v) \) and \( \langle Bu, v \rangle_{m,2} = \int_G f(u, x)v \). Noticing that both inner products are linear in \( v \), we are able to apply Riesz' representation theorem for linear functionals on the Hilbert space \( W_{m,2}(G) \) to obtain the required operator equation. The inequality satisfied by \( A \) is a consequence of Gårding's Inequality, while the properties of \( B \) follow from Sobolev's Imbedding Theorem.

Set \( F(t, x) = \int_s^t f(s, x)ds \).

DEFINITION 3. \( \partial M_R \) is the set of all functions \( u(x) \) such that:

(a) \( u(x) \in W_{m,2}(G) \),
(b) \( \int_{\partial G} F(u, x) = R \).

(We refer to \( \partial M_R \) as the energy level with radius \( R \)).
Lemma 1 (Geometry of Energy Levels). Let $R$ be a fixed positive number. Then

(i) $\partial M_R$ contains elements of $\mathcal{W}_{m,2}(G)$.

(ii) $\partial M_R$ is weakly closed and, on $\partial M_R$, $\|u\|_{m,2} \leq c(R) > 0$, for some constant $c(R)$ independent of $u$.

(iii) On $\partial M_R$, $\|u\|_{0,1} \leq g(R)$, where $g(R)$ is a constant independent of $u$.

Theorem II (Existence Theorem). Let $G$ be any bounded domain $\mathbb{R}^n$. Suppose $f(t, x) \in Z(p_1)$. Then the boundary-value problem (1) has a generalized eigenfunction $u(x)$; $u(x)$ is normalized by the requirement that $u(x) \in \partial M_R$ for some fixed positive $R$ and characterized as a solution of the variational problem $\inf a(v, v)$ for $v \in \partial M_R$.

This result is proved by solving the above variational problem by the direct method of the calculus of variations, using the geometrical properties of $\partial M_R$, as illustrated in Lemma 1. We then show that a solution of the variational problem is also a solution of the operator equation.

We note the following regularity conditions associated with the boundary-value problem (1): (1a) $G$ is of class $C^m$. (1b) For the coefficients of $A$, $a_{\alpha\beta}(x) \in C^{m}(\overline{G})$, (1c) $f(t, x)$ satisfies a local Lipschitz condition in $t$ and a local Hölder condition of exponent $r$, $0<r<1$, for $x \in G$.

Theorem III (Regularity Theorem). Suppose the regularity conditions (1a), (1b), (1c) for the boundary-value problem (1) hold, and $f(t, x) \in Z(p_1)$. Then any generalized eigenfunction for (1) is a classical eigenfunction.

This result follows immediately from the results of Agmon [1], Agmon, Douglis and Nirenberg [2] and Browder [3], [4] on the $L_p$ regularity theory of elliptic operators.

Theorem IV (Positive Eigenfunctions). Let the hypotheses of Theorem II be satisfied. Then if $A$ is a second-order operator, the boundary-value problem (1) has a positive generalized eigenfunction $u(x)$ in $G$. If $u(x)$ is a classical eigenfunction, then $u(x) > 0$ in $G$.

Remark. The boundary-value problem studied here provides another example of nonuniqueness in the theory of quasi-linear elliptic equations of the type similar to the Navier-Stokes equation for a stationary flow of an incompressible fluid.

Added in proof (December 7, 1964). A variational method can also be used to prove the existence of an infinite number of distinct nor-
malized eigenfunctions $u_n$ with associated eigenvalues $\lambda_n$, tending to infinity with $n$, for the boundary value problem (I).

**BIBLIOGRAPHY**


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