THE KERVAIRE INVARIANT OF $(8k+2)$-MANIFOLDS

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1. **Statements of results.** Let $\Omega_m(\text{Spin})$, $\Omega_m(SU)$, and $\Omega_m(e)$ denote the mth spinor, special unitary, and framed cobordism groups respectively (see [6]). In [4] Kervaire defined a homomorphism $\Phi: \Omega_{2n}(e) \to Z_2$ for n odd and $n \neq 1$, 3, or 7, and showed that $\Phi = 0$ for $n = 5$. Kervaire and Milnor state in [5] that $\Phi = 0$ for $n = 9$. One of the corollaries of our results is that $\Phi = 0$ for $n = 4k+1$, $k \geqq 1$.

In [2] a homomorphism $\Psi: \Phi_{2n}(\text{Spin}) \to Z_2$ was defined for $2n = 8k+2$, $k \geqq 1$, such that $\Phi = \Psi\rho$, where $\rho: \Omega_{2n}(e) \to \Omega_{2n}(\text{Spin})$ is the obvious map. Ψ induces a map from $\Omega_{2n}(SU)$ into Z_2 which we also denote by Ψ. It is easily verified that $\Omega_1(SU) = \Omega_1(\text{Spin}) = \Omega_1(e) = Z_2$. Let α be the generator. Let θ be the secondary cohomology operation coming from the relation $Sq^2Sq^2 = 0$ on an integer cohomology class [7]. If f is a map, let θ_f denote the associated functional cohomology operation [7].

The main theorems of this announcement are the following.

THEOREM 1.1. *If $\beta \in \Omega_{8k}(\text{Spin})$ and $k \geqq 1$, then $\Psi(\alpha^2 \cdot \beta) = \chi(\beta)$, where $\chi(\beta)$ is the Euler characteristic of β reduced* mod 2.

THEOREM 1.2. *If $\beta \in \Omega_{8k}(SU)$ and $k \geqq 1$, then $\Psi(\alpha^2 \cdot \beta) = \theta_\nu(v^2)(M)$, where M is a 3-connected manifold representing $\alpha^2 \cdot \beta$, $\nu: M \to BSU$ is the classifying map of the SU-structure on the normal bundle of M, and $v \in H^{4k}(BSU; Z)$ is such that v reduced* mod 2 *is v_{4k} in the expression $\overline{W} = Sq(1 + v_2 + v_4 + \cdots)$.*

We now deduce some corollaries of these two theorems.

COROLLARY 1.3. *$\Phi: \Omega_{8k+2}(e) \to Z_2$ is zero if $k \geqq 1$.*

PROOF. Conner and Floyd [9] and Lashof and Rothenberg[2] have shown that if $\gamma \in \Omega_{8k+2}(SU)$ goes to zero in $\Omega_{8k+2}(U)$, then $\gamma = \alpha^2 \cdot \beta$, where $\beta \in \Omega_{8k}(SU)$. In particular, if $\gamma = \bar{\rho}(\delta)$, $\delta \in \Omega_{8k+2}(e)$, then $\gamma = \alpha^2 \cdot \beta$. Let $\delta = [M]$. Then M can be taken to be 3-connected and ν is homotopic to a constant. The corollary now follows from Theorem 1.2.

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COROLLARY 1.4. *bP_{8k+2}, the group of homotopy spheres bounding $(8k+2)$-dimensional π-manifolds, is isomorphic to Z_2 if $k\geqq 1$.*

PROOF. See [5, p. 536].

COROLLARY 1.5. *If M^{8k+2} is the closed combinatorial manifold obtained by plumbing together two copies of the tangent disc bundle of S^{4k+1} and attaching on $(8k+2)$-disc, then M^{8k+2} does not have the homotopy type of a differentiable manifold.*

PROOF. See [4].

COROLLARY 1.6. *Any element of $\Omega_{8k+2}(e)$, $k\geqq 1$, can be represented by a homotopy $(8k+2)$-sphere.*

PROOF. See [4].

The final corollary is in a different direction.

COROLLARY 1.7. *There exists $\gamma\in\Omega_{8k+2}(\mathrm{Spin})$ and $\delta\in\Omega_{18}(SU)$ such that $\Psi(\gamma)=\Psi(\delta)=1$. Furthermore, γ and δ can be chosen to be orientably cobordant to zero and hence Ψ cannot be factored through $\Omega_{8k+2}=\Omega_{8k+2}(SO)$.*

PROOF. Let $\gamma=\alpha^2\cdot\beta$, where β is a product of even-dimensional quaternionic projective spaces and use Theorem 1.1. By an analysis of the Adams spectral sequence for MSU, one may show that there is a $\beta'\in\Omega_{16}(SU)$ with $\chi(\beta')=1$. Let $\delta=\alpha^2\cdot\beta'$ and use Theorem 1.1.

2. **Outline of proofs.** Throughout this section all cohomology groups have Z_2 coefficients and $n=4k+1$, $k\geqq 1$. We will only consider SU-cobordism because the cohomology operations are simpler in this case; the spinor case of Theorem 1.1 is a fairly easy generalization. All manifolds will be assumed to have an SU-structure on their normal bundle and if M is such a manifold, then $\nu(M)\colon M\to BSU(q)$, $T(M)\colon T(\nu)\to MSU(q)$, and $f(M)\colon S^{m+2q}\to MSU(q)$ will denote, respectively, the classifying map, the associated map on the Thom spaces, and the Thom construction associated to the SU-structure. S^1 will have the SU-structure so that $[S^1]=\alpha\in\Omega_1(SU)$.

In [3] it is shown how the relation $Sq^2Sq^{n-1}+Sq^1(Sq^2Sq^{n-2})=Sq^{n+1}$ leads to a quadratic cohomology operation

$$\phi\colon H^n(X)\cap \operatorname{Ker} Sq^{n-1}\cap \operatorname{Ker} Sq^2Sq^{n-2} \to H^{2n}(X)/Sq^1(H^{2n-1}(X)) + Sq^2(H^{2n-2}(X)).$$

If M is a closed $2n$-manifold with Stiefel-Whitney classes $W_1(M)=0$ and $W_2(M)=0$, then $\phi\colon H^n(M)\cap\operatorname{Ker} Sq^{n-1}\to H^{2n}(M)$. We define a function $\lambda\colon\Omega_{2n}(K(Z_2,n);SU)\to Z_2$ as follows. If $u\in H^n(M)$ and

$Sq^{n-1}(u)=0$, then $\lambda(\{M, u\})=\phi(u)(M)$. If $Sq^{n-1}(u)\neq 0$, we change (M, u) in its bordism class so that $Sq^{n-1}(u)=0$, e.g. make M simply connected. Then $\Psi(M)=\sum\lambda(\{M, u_i\})\cdot\lambda(\{M, v_i\})$, where $\{u_i, v_i\}$ is a symplectic basis for $H^n(M)$ (see [2] and [3] for details of ϕ and [5] for details of symplectic bases).

Let N be a 1-connected closed $(2n-2)$-manifold. Let P be 1-connected and M be 3-connected with $[P]=\alpha\cdot[N]$ and $[M]=\alpha^2\cdot[N]$. Let η be an appropriate suspension of the Hopf map from S^3 to S^2. Note that $f(M)\simeq f(N)\eta\eta$ and $f(P)\simeq f(N)\eta$. Let $x\in H^1(S^1)$ be the generator, let ${}^1\phi$ be the suspension of ϕ, and let U_q be the Thom class of whatever Thom space is appropriate.

Theorem 1.2 is proved by verifying each of the following "equalities." In each case, the indeterminacy is zero and the element described lies in a group isomorphic to Z_2.

$$\begin{aligned}\Psi(N\times S^1\times S^1) &= \lambda(N\times S^1\times S^1, \nu(N)^*(v)\otimes 1\otimes x)\\ &= \lambda(P\times S^1, \nu(P)^*(v)\otimes x) = \phi(\nu(P)^*(v)\otimes x)\\ &= {}^1\phi(\nu(P)^*(v)) = Sq^2_{\nu(P)}(v^2) = Sq^2_{\nu(P)}(v^2)\cdot U_q\\ &= Sq^2_{T(P)}(v^2\cdot U_q) = Sq^2_{f(P)}(v^2\cdot U_q)\\ &= Sq^2_\eta(f(N)^*(v^2\cdot U_q)) = Sq^2_\eta Sq^2_\eta(f(N)^*(v^2\cdot U_q))\\ &= \theta_{\eta\eta}(f(N)^*(v^2\cdot U_q)) = \theta_{f(M)}(v^2\cdot U_q)\\ &= \theta_{\nu(M)}(v^2)\cdot U_q = \theta_{\nu(M)}(v^2).\end{aligned}$$

The first equality is obtained from a careful choice of a symplectic basis for $N\times S^1\times S^1$. The fifth, seventh, eleventh, and fourteenth equalities follow respectively from results in [8], [1], [7], and [1]. Theorem 1.1 follows from the equation $\Psi(N\times S^1\times S^1)=Sq^2_\eta(f(N)^*(v^2\cdot U_q))$ and the facts that Sq^2_η is an isomorphism and $\nu(N)^*(v^2)=\chi(N)$.

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