BOOK REVIEWS

Introduction to quadratic forms. By O. T. O'Meara. (Die Grundlehren der mathematischen Wissenschaften, Vol. 117.) Springer, Berlin; Academic Press, New York; 1963. xi+342 pp. $12.00.

The arithmetic theory of quadratic forms, one of the most venerable subjects of present-day mathematics, may well appear to consist of a collection of scattered results from elementary number theory on such questions as the representation of integers by squares—at least this is the impression one obtains after reading Dickson's *History*, for example. Thus, completely apart from any other merits O'Meara's *Introduction to quadratic forms* may possess or lack, its appearance should cause pleasant surprise among many classically-trained mathematicians, not excluding quadratic formalists themselves, who will find that so orderly and unified a body of material can be presented from such an apparent assortment of facts.

This unification is not, however, the only novelty of the book. O'Meara, assuming as prerequisite a knowledge of first-year algebra and elementary topology, presents his subject within the framework of abstract algebra, thereby providing both greater clarity and generality; for example, p-adic numbers enter as a natural outgrowth of valuation theory, rather than as the formal power series with complicated rules of operation used by Jones in *The arithmetic theory of quadratic forms*, or as the frightful systems of congruences modulo prime powers used by Watson in his *Integral quadratic forms*. Furthermore, being able to quote from linear algebra, O'Meara can use the language of inner-product spaces, thus avoiding the cumbersome double-summations of the forms themselves. But the most striking point of all is the author's realization that the only suitable way to approach the subject, even in the case of forms having rational coefficients, is via algebraic number theory, a fact he exploits to the fullest.

In this "modern" approach, O'Meara is following the example set by Eichler in *Quadratische Formen und orthogonale Gruppen*, but the two books are hardly comparable, even apart from the discoveries of the eleven years between their dates of publication. Eichler has gone less deeply into matters of classification and has instead included analytic theory; he has written a work that is extremely condensed and hardly self-contained, neither of which can be said for the book under review. It should be hurriedly pointed out that the words "modern" and "abstract" are here being used without praiseworthy or derogatory connotations: an author's task should be to present his material in the setting most understandable to his contemporaries, and so it is not necessarily a virtue to write in the most elementary manner nor, as is apparently the fashion these days, in the most abstruse. In any event, it is certainly evident that O'Meara has tried to assist his readers rather than his own ego.

As was mentioned above, algebraic number theory plays a major role in this approach to the subject—there are in fact few instances in mathematics in which one discipline seems so well suited to the study of another—and so the first three chapters are devoted to an exposition of those parts of algebraic number theory which are to be used later. The treatment of this preliminary material is quite thorough and, were it not for the sudden appearance during the past year of several books dealing wholly with algebraic number theory, would have made the ideal introduction to the valuation-theoretic approach to that theory. There are many excellent touches in these three chapters, such as the definition, in Chapter I (a study of rank 1 valuations), of *a*, rather than *the*, residue class field, using a universal mapping property—a minor point, surely, but one which avoids a spot sore for many students; and the axiomatic treatment of Dedekind Ideal Theory in Chapter II is, to the reviewer's knowledge, unexcelled anywhere in the literature. Chapter III introduces the concept of an idèle, and proves in quick succession, à la Artin-Whaples, three major theorems for global fields (i.e., algebraic number fields, and algebraic function fields in one indeterminate over finite constant fields): the product formula, the Dirichlet unit theorem, and finiteness of class number for ideals.

In Chapter IV first appears a quadratic form, in the guise of the quadratic space (a finite-dimensional vector space, over a field of characteristic $\neq 2$, with a given symmetric "scalar product"). The geometric language for studying these spaces is introduced, and various general theorems are proved, such as Witt's cancellation law. Then the orthogonal group is considered, with particular attention given to certain of its subgroups, in the manner of the work of Dieudonné.

Chapter V, "The Algebras of Quadratic Forms," continues the study of quadratic forms over general fields. Evidently feeling that the theory of finite-dimensional associative algebras is no longer standard fare in a first-year algebra course, O'Meara has inserted a detailed treatment of central simple algebras and their tensor products, even giving a proof of the pertinent Wedderburn structure theorem that differs from the usual ones in that idempotents are avoided completely (not necessarily an improvement in itself: we feel that here is one instance in which the older, computational proof, such as the one in Albert's *Structure of algebras*, is clearer and more instructive than the "conceptual" one). This is immediately applied in the study of the Hasse algebra, which occurs later as one of the arithmetic invariants, and the Clifford algebra, used to define the spinor norm.

The application of algebraic number theory to quadratic forms begins in Chapter VI. The fundamental question there is the determination of invariants for isometry of two quadratic spaces, or fractional equivalence of two quadratic forms in the older terminology. This chapter is probably the high point of the book, certainly the most satisfying aesthetically, for complete answers are given in the case of the fields arising in algebraic number theory, i.e., the real and complex numbers, finite fields, local fields, such as the p-adic numbers, and the above-mentioned global fields, where the celebrated Hasse-Minkowski theorem holds: two spaces over a global field are isometric if and only if they are locally isometric at every prime spot. O'Meara's book, incidentally, is the only one that gives a self-contained proof of this theorem, done by specializing class field theory to the case of quadratic extensions. Even for spaces over the rational numbers, Jones and Watson appeal to the Dirichlet theorem on primes in arithmetic progressions at a crucial point in the argument.

Chapter VII continues with the global questions raised in Chapter VI, in particular, the existence of forms over algebraic number fields having prescribed invariants; one regrets that the parallel problem for function fields was not also included, and a word of explanation for the statement "The formula is actually true over any global field, but we shall not go into the function theoretic case here," on p. 190, should at least have been given. The chapter concludes with a proof of the Legendre-Gauss law of quadratic reciprocity; methods of abstract algebra are rapidly incorporating large parts of elementary number theory into a rational-number-field case of algebraic number theory while relegating the rest of it to the high school math club (perhaps rightly so), and O'Meara is not the only recent author unable to resist the temptation of showing how Gauss' famous bit of labor drops out of more abstract considerations in a very few lines.

The remaining three chapters of the book deal with the integral theory. In the older terminology of forms, the main problem is this: given two forms in a local or global field F, find invariants for their equivalence with respect to the ring of integers of F. Historically, such integral questions were the earliest to be considered, and only later was it realized that their solution first required that of fractional equivalence.

Perhaps the most original part of the book is Chapter VIII, which presents a general formulation of the problem of equivalence of forms over a Dedekind domain, using the geometric concept of a lattice (which in this context denotes a subset of a quadratic space, over the quotient field of the given domain, that is at the same time a module with respect to the domain). This chapter is a sort of integer-analogue of Chapter IV: a linear algebra for lattices is set up, and various questions relating quadratic forms and lattices are studied.

Arithmetic questions return in Chapter IX, where the local integral invariants are derived; here again the reader can have the satisfaction of seeing a problem raised and then completely solved, and might find the material even more interesting upon realizing that the solution of this problem constitutes the author's earliest published research. The invariants obtained, although formidable at first glance, are introduced in a natural and well-documented manner, and the present treatment represents a considerable improvement in readability over that of the original papers. The chapter ends with a study of the orthogonal group over a local field, where the arithmetic structure provides much more information than can be obtained over a general field.

The final chapter, "Integral Theory of Quadratic Forms over Global Fields," is the closest in spirit to the classical theory, although it is of course handled in the same abstract setting as the preceding chapters, and includes only a small fraction of the specialized information on, say, rational integral binary forms contained in Dickson or even Jones (the eager reader might find it well worth his while to rewrite—and attempt to generalize—the various results in integral theory that appear in Jones' book); on the other hand, O'Meara is working much more generally over number and function fields, whereas the other two authors restrict themselves throughout to the rational numbers, where more specific answers can be given. The recently-discovered spinor genus of Eichler and Kneser is introduced, the clearest treatment of it yet to appear in print; it will be recalled that the spinor genus gives a classification of forms intermediate to those afforded by the class and the ordinary genus, and has proven of fundamental importance in the global integral theory. Finiteness of class number for forms is proven. Finally, in the very last paragraph, the author unbends to the extent of classifying all unimodular lattices of dimension at most 9 in a definite space over the rational numbers.

Anyone who has heard O'Meara lecture will recognize in every page of this book the crispness and lucidity of the author's style; the reader soon comes to feel himself cheated when he finds a proof requiring an extra sentence or two of explanation. Furthermore, the organization and selection of material is superb: hardly any result is proven early in the book without its being used later on, and even the so-called "Examples" frequently have application in later arguments. The author's relentless pace imparts to the book almost the flavor of a research paper (although with complete proofs), certainly not of a leisurely textbook.

And it is just this relentlessness that we find to be the sole defect of the volume, for there is practically no motivation anywhere; this was most evident through the reviewer's own reactions to the different topics studied, an "Of course! how beautiful!" to a familiar subject, a "So what?" to a new one. In the hands of a teacher having some knowledge of the field, the book is eminently suitable as a course-text, for the missing motivation is virtually begging to be heard, and the lecturer has in it a ready-made vehicle for his own brilliant performance. But the reader approaching quadratic forms for the first time, and thus required to accept on authority that "all will turn out right at the end"—as indeed it does—may find its 335 pages of solid mathematics a severe challenge to his faith. It should not have required many additional pages to insert an occasional preview of coming attractions, and it is to be regretted that the author did not choose to do this.

Except for the absence of motivation, however, *Introduction to quadratic forms* deserves high praise as an excellent example of that too-rare type of mathematical exposition combining conciseness with clarity; many a current or prospective writer could do far worse than to choose it as his model. Considering the sparseness of available literature on the subject, the author has surely done a distinct service to the mathematical community by writing his book. Incidentally, typography is at the usual high level to be expected in the Springer Yellow Series, and the few misprints that occur are easily recognized and corrected.

It would be interesting to speculate on the subject O'Meara will next choose to unify and expound upon in his own elegant style.

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